A unifying approach to the regularization of Fourier polynomials

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In a previous paper [4] the following problem was considered: find, in the class of Fourier polynomials of degree n, the one which minimizes the functional:

\[ J^*[F_n, \alpha] = \| f - F_n \|^2 + \sum_{r=1}^{\infty} \frac{\sigma_r^2}{r!} \| F_n^{(r)} \|^2, \]  

where \( \| \cdot \| \) is the \( L^2 \) norm, \( F_n^{(r)} \) is the rth derivative of the Fourier polynomial \( F_n(x) \), and \( f(x) \) is a given function with Fourier coefficients \( c_k \).

It was proved that the optimal polynomial has coefficients \( c_k^* \) given by:

\[ c_k^* = c_k e^{-\alpha k^2}; \quad k = 0, \pm 1, \ldots, \pm n. \]  

In this paper we consider the more general functional:

\[ J[F_n, \sigma_r] = \| f - F_n \|^2 + \sum_{r=1}^{\infty} \sigma_r \| F_n^{(r)} \|^2, \]  

which reduces to (0.1) for \( \sigma_r = \sigma^r/r! \).

We will prove that the classical sigma-factor method for the regularization of Fourier polynomials may be obtained by minimizing the functional (0.3) for a particular choice of the weights \( \sigma_r \). This result will be used to propose a motivated numerical choice of the parameter \( \sigma \) in (0.1).

1. Introduction

It is well known [8,6] that the Gibbs phenomenon arises when we approximate in the interval \([-\pi, \pi)\) a discontinuous periodic function \( f(x) \) of period \( 2\pi \) by using Fourier polynomials.

Some techniques have been proposed in the past in order to reduce the Gibbs phenomenon. Among these, we consider here the sigma-factor method which starts from the idea of considering the continuous function \( f_c(x) \) obtained by the convolution of \( f(x) \) with the function \( p(x) \) defined as

\[ p(x) = \frac{1}{2\pi} \left( 1 - \cos \frac{x}{2} \right). \]
\[ p(x) = \begin{cases} (2\pi)^{-1}, & |x| \leq \pi, \\ 0, & |x| > \pi, \end{cases} \] (1.1)

and of taking as approximation of \( f(x) \) the Fourier polynomial of degree \( n \) of \( f_c(x) \).

By the convolution theorem it is easy to show [7] that the Fourier coefficients \( c_k \) of \( f_c(x) \) are given by

\[ c_k = \frac{\sin(k\tau)}{k\tau}; \quad k = 0, \pm 1, \pm 2, \ldots \] (1.2)

Indeed, see [7], \( P(\omega) = (\sin \omega \tau)/(\omega \tau) \) is the Fourier transform of the function (1.1).

In fig. 1 we can see, as an example, the step function \( f \) and the related function \( f_c \).

More recently, in [2], a new approach for the regularization of Fourier polynomials was proposed, based on the minimization of the following functional:

\[ J_2[F_n, \sigma] = \|f - F_n\|^2 + \sigma_2 \|F_n^{(2)}\|^2, \] (1.3)

where \( \sigma_2 > 0 \) is a regularization parameter and \( F_n^{(2)} \) is the second derivative of \( F_n \).

In [4] the functional (0.1), where the derivatives have weights \( \sigma_r = \sigma^r/r! \), was considered. It is easy to see that this functional (0.1) and (1.3) are particular cases of the functional (0.3).

In the next section we will prove that the sigma-factor technique may be obtained by the minimization of (0.3) for a particular choice of \( \sigma_r \), so that the functional (0.3) may be seen as a unifying regularization functional. Moreover, we will prove that the coefficients \( c_k^* \) defined in (0.2) are the Fourier coefficients of the function given by the convolution of \( f(x) \) with the function \( g(x) \) defined by

\[ g(x) = \frac{1}{2\sqrt{\sigma \pi}} e^{-x^2/(4\sigma)}. \] (1.4)

Fig. 1. The step function \( f \) (full line) and the regularized function \( f_c \) (dotted line).