SEMISYMMETRIC SUBMANIFOLDS

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This article surveys new work on semisymmetric and k-parallel submanifolds $M^m$ in $E^n$ and $M^n(c)$.

§1. INTRODUCTION. OBJECTS OF INTEREST

Semisymmetric submanifolds $M^m$ in Euclidean spaces $E^n$ or in spaces of constant curvature $M^n(c)$ are natural generalizations of symmetric submanifolds. They are the geometric analogs of semisymmetric Riemann spaces and are obtained by immersions of the latter, just as symmetric submanifolds are the geometric analog of symmetric Riemann spaces.

As we know, locally symmetric spaces are characterized by the admission, in some neighborhood of any point, of geodesic symmetry relative to this point. Their analytic indication is the condition $\nabla R = 0$. An integrability condition for this system of differential equations is representable in the form $R(X,Y) \circ R = 0$, where $X$ and $Y$ are arbitrary vectors in the tangent space $T_x M^m$ and $R(X,Y)$ is the corresponding curvature operator, which acts linearly on each argument of the tensor $R$, forming the sum of the four transformed tensors thus obtained. Spaces for which this integrability conditions is satisfied were introduced as early as the 1920's by P. A. Shirokov and E. Cartan and were later named semisymmetric Riemann spaces \[25, 26, 1, 2\]. They form, of course, a subclass of the class of locally symmetric spaces.

Semisymmetric Riemann spaces attracted attention after Nomizu's hypothesis \[65\] (completely irreducible semisymmetric spaces are locally symmetric) was proved false \[75, 81\] by construction of a hypersurface with $R(X,Y) \circ R = 0$ and $\nabla R \neq 0$ (see also \[79\]). Their connection with Lie triple systems was established in \[3\], and their image under geodesic mappings was explained in \[25, 26, 62\]. Semisymmetric Riemann spaces were classified in \[78, 80\]. It turns out that under certain additional assumptions, the Nomizu conjecture is true \[40\].

Locally symmetric submanifolds $M^m$ are characterized as admitting, in some neighborhood of any point $x \in M^m$, symmetry of the enveloping space $M^n(c)$ with respect to normal $(n - m)$-planes (i.e., totally geodesic submanifolds) at this point $x$. It is clear that they are obtained by certain isometric imbeddings of locally symmetric Riemann spaces, but they do not exhaust all such immbeddings. This was the subject of, for example, \[22, 43\].

It was established in \[29, 39, 77\] that the analytic test for local symmetry is the condition $\bar{\nabla} h = 0$, where $\bar{\nabla} = \nabla \oplus \nabla \perp$ is the van der Waerden-Bortolotti connection (here $\nabla \perp$ is the normal connection of the submanifold $M^m$ in $M^n(c)$). The analytic condition $\bar{\nabla} h = 0$ (parallelness of the second fundamental form $h$ in the connection $\bar{\nabla}$) is frequently used as a starting point, and submanifolds satisfying the condition are said to be parallel \[82, 84, 86\].

Various properties of submanifolds with $\bar{\nabla} h = 0$ in Euclidean spaces were observed even before their symmetry was established. It was shown in \[87\] that they have totally geodesic Gauss images; it should be added that all submanifolds with this property were found in \[30, 69\] (see also \[12\]), and all submanifolds with totally umbilical Gauss images were found in \[44\].

Of the submanifolds with $\bar{\nabla} h = 0$, the first classified were those with flat normal connection. It was shown in \[89, 88\] that all of them are the product of a sphere and a plane. This result was obtained earlier for the special cases of hypersurfaces \[76\] and submanifolds of codimension 2 \[90\]. Pseudoumbilical surfaces ($m = 2$) with $\bar{\nabla} h = 0$ were classified in \[41\].

The general problem of classifying submanifolds with $\bar{\nabla} h = 0$ (i.e., locally symmetric submanifolds) was solved by Ferus \[36-39\], who showed that they are the products of their irreducible components, which prove to be symmetric R-spaces in standard imbeddings and the minimal submanifolds of certain spheres. The different properties of the imbeddings of such R-spaces are discussed in \[46, 83, 19, 45, 68\].
The study of submanifolds with $Vh = 0$ in spaces of constant curvature was begun in [13], and ended with [82, 29], which provided a complete classification complementing Ferus’. The authors of [29] used the connection between symmetric submanifolds and Jordan triple systems that was implicitly used in [39] but not explicitly used until [28].

The condition for integrability of the system $Vh = 0$, in virtue of the known identity $R(X, Y) \circ h = VxVylh$, is $R(X,Y) \circ h = 0$. Submanifolds satisfying this identity are said to be semisymmetric [48-51] (or semiparallel [31-33]); their class includes the locally symmetric (i.e., parallel) submanifolds. In view of the above-noted identity, the class of semisymmetric submanifolds is characterized by the condition $V(X)V(Y)h = 0$, which can be interpreted in the language of higher-order fundamental forms.

Namely, if we write $\tilde{V}h = \tilde{V}h$, and we say that this is the $(k + 2)$-th order fundamental form of the submanifold $M^m$, the condition for semisymmetry is equivalent to the requirement that the 4-th order fundamental form be symmetric in all of its arguments; recall that, in virtue of the Peterson–Codazzi equation $Vxh(Y, Z) = Vyh(X, Z)$, the third fundamental form (i.e., the third order form) of $Vh$ is always symmetric. This permits us to distinguish one class of semisymmetric manifolds immediately — the subclass of submanifolds with parallel third fundamental form (or 2-parallel submanifolds), which are characterized by the condition $\tilde{V}Vh = 0$, i.e., the condition $\tilde{V}^2h = 0$.

Generally, if $\tilde{V}^k h = 0$, $k \geq 1$, we speak of submanifolds with $(k - 1)$-th order parallel fundamental forms; more briefly, we call them k-parallel submanifolds; for $k = 1$, they are locally symmetric.

It follows immediately from Gauss’ formula relating $R$ and $h$ that $\tilde{V}^k h = 0$ implies $v^{2k-1}R = 0$, which Nomizu (and Ozeki, see [3], Remark 7) showed implies $v R = 0$, i.e., it implies internal local symmetry of the submanifold. Thus, k-parallel submanifolds $M^m$ are obtained as the result of certain special imbeddings of locally symmetric Riemann spaces in $M^n(c)$. For $k = 2$, they are semisymmetric (as submanifolds), but the situation is still not clear for $k > 2$ (see also §6).

One very important class of semisymmetric submanifolds $M^n$ in $E^n$ is the recurrent submanifolds, which were introduced in [33]. A submanifold $M^n$ is said to be recurrent if it is possible to construct a 1-form $\mu$ on it so that $\tilde{V}xh(Y, Z) = \mu(X)h(Y, Z)$. It was proved in [33] that then $M^n$ is either a locally symmetric submanifold or the product of 2-plane lines and an $(m - 1)$-plane.

This exhausts the general subclasses of semisymmetric submanifolds that have been investigated. It remains to note that locally symmetric submanifolds have been investigated both in enveloping spaces other than Euclidean space $E^n$ and spaces of constant curvature $M^n(c)$, and in connection with other properties [47, 59-61, 63, 64, 70, 71, 84-86]. Our survey is limited to the cases in which the more general enveloping space can be imbedded in $E^n$ or $M^n(c)$ as a symmetric submanifold.

This survey is devoted to semisymmetric and h-parallel submanifolds $M^m$ in $E^n$ or $M^n(c)$. We will consider recent results that are still available only in individual articles, frequently in publications that are difficult to obtain. We are concerned primarily with problems of classification and geometric description of the submanifolds under consideration. We will also present some unpublished or only announced results. We present theorems in §4 on the structure of normally flat semisymmetric submanifolds $M^m$ in $E^n$ that complement the results of [9], is an exception, in that it contains rather complete proofs.

§2. GENERAL PROPERTIES OF SEMISYMMETRIC SUBMANIFOLDS

First of all, we will present more detailed analytic versions of the definitions and statements made above. For this purpose we will use the formalism of a bundle of adapted orthonormal frames $O(M^m,M^n(c))$, i.e., coordinate systems $\{x; e_1, \ldots, e_m; e_{m+1}, \ldots, e_n\}$, where $x \in M^m$, $e_i \in T_x M^m$, $e_x \in T_{x-1} M^m; i, j, \ldots = 1, \ldots, m; \alpha, \beta, \ldots = m + 1, \ldots, n$.

Let $\omega = J \otimes e_1$ be a canonical 1-form on $M^n(c)$ and let $\omega^k_j$ be the forms of the Levi-Civita connection in $O(M^n(c))$; $J, K, \ldots = 1, \ldots, n$. Then $\omega^K_j + \omega^J_j = 0$, $d\omega^I_j = \omega^K_j \wedge \omega^K_j$, $d\omega^I_j = -\omega^J_j \wedge \omega^K_j + \omega^K_j \wedge \omega^K_j$ and for the adapted frames $\omega^a = 0 \Rightarrow \omega^a \wedge \omega^a = 0 \Rightarrow \omega^a = h_{ij}^a \omega^i \wedge \omega^j = 0$ (where $h_{ij}^a = dh_{ij}^a - h_{kj}^a \omega^k - h_{kj}^a \omega^k$

Here $h_{ijk}^a$ is defined the same way as $h_{ij}^a$; $\Omega h_{ij}^a = h_{ij}^b \Omega^b + h_{ij}^c \Omega^c - h_{ij}^b \Omega^b$, $\Omega^b = -R^b_{pe} \omega^p \wedge \omega^q$, $R^a_{pe} = c_{ij}^a \delta^a_1 + h_{ij}^a \delta^a_1$, $R^a_{pe} = h_{ij}^a \delta^a_1$. 1610