BRIEF COMMUNICATIONS

ON INFINITE NONTANGENTIAL BOUNDARY VALUES OF
FUNCTIONS, SUBHARMONIC IN A LIPSCHITZ DOMAIN

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The theorem of N. N. Luzin and I. I. Privalov [1] asserts that a function \( f \neq 0 \), holomorphic in the unit circle, can have nontangential boundary values equal to zero only on a set of zero measure. Introducing the subharmonic function \( u = \ln |f| \), we obtain an equivalent statement: the set of the points of the circumference, where the nontangential boundary values of the function \( u \) are equal to \(-\infty\), has zero measure. A conformal mapping enables us to obtain an analogous result for domains other than a circle.

A similar theorem for subharmonic functions in the semispace \( \mathbb{R}^{n+1}_+ \) is proved in [2]. The Kelvin transform proves this theorem for the unit ball in \( \mathbb{R}^{n+1} \). Below we give the proof for Lipschitz domains.

An open bounded domain \( \mathcal{D} \) in the Euclidean space \( \mathbb{R}^{n+1} \) is said to be a Lipschitz domain if for some \( c > 0 \) for each point \( Q \in \partial \mathcal{D} \) (the boundary of \( \mathcal{D} \)) there exists a neighborhood \( V(Q) \), a system of coordinates \( (x, y) \), where \( x \in \mathbb{R}^n \), \( y \in \mathbb{R} \), and a function \( \varphi \) such that

1) \( |\varphi(x_1) - \varphi(x_2)| \leq c|x_1 - x_2| \) for any \( x_1 \) and \( x_2 \);
2) \( V(Q) \cap \mathcal{D} = V(Q) \cap \{ (x, y): y > \varphi(x) \} \);
3) \( V(Q) \cap \partial \mathcal{D} = V(Q) \cap \{ (x, y): y = \varphi(x) \} \).

One can assume that the neighborhood \( V(Q) \) is cylindrical in these local coordinates: \( Q = (x_0, y_0) \)

\[ V(r, \delta, Q) = \{(\xi, \eta): |\xi - x_0| < r, |\eta - y_0| < \delta\} \]

and \( |\varphi(\xi) - y_0| < cr \leq \delta/2 \). One can select the parameters \( r_0 \) and \( \delta_0 \) of the cylinder to be the same for all the points \( Q \in \partial \mathcal{D} \) so that the bottoms of the cylinder \( V(r_0, \delta_0, Q) \) do not touch \( \partial \mathcal{D} \).

A cone \( \Gamma(h, \rho, Q) \) with vertex at the point \( Q = (x, y) \in \partial \mathcal{D} \), defined in the local system of coordinates by the equality

\[ \Gamma(h, \rho, Q) = \{ (\xi, \eta): \rho|\xi - x| < \eta - y < h \} \]

is said to be a nontangential interior cone if there exist \( h_1 > h \) and \( \rho_1 < \rho \) such that \( \Gamma(h_1, \rho_1, Q) \subset \mathcal{D} \).

A function \( u: \mathcal{D} \to \mathbb{R} \) has at a point \( Q = (x, y) \in \partial \mathcal{D} \) a nontangent boundary value \( L \) if the restriction to any nontangential cone \( \Gamma(h, \rho, Q) \) has limit \( L \) at the point \( Q \). Notation:

\[ \lim_{P \to Q} u(P) = L. \]

The harmonic measure of a set \( E \subset \partial \mathcal{D} \) relative to the domain \( \mathcal{D} \) at the point \( P \) will be denoted by \( \omega(P, E) \). The measures \( \omega(P_1, E) \) and \( \omega(P_2, E) \) are mutually absolutely continuous.

R. A. Hunt and R. L. Wheeden [3, 4] have proved the following lemmas.

**LEMMA 1.** If \( \mathcal{D} \) is a Lipschitz domain and \( \Delta(Q, r) = \partial \mathcal{D} \cap V(r, \delta, Q) \), then there exists a constant \( a > 0 \) such that \( \omega(P, \Delta(Q, 2r)) \geq a \) for \( P \in V(r, rs, Q) \), where \( r \leq r_0/2, rs \leq \delta_0; a \) depends only on \( s \) and \( \mathcal{D} \).

**LEMMA 2.** If \( \mathcal{D} \) is a Lipschitz domain, \( E \) is a Borel subset of \( \partial \mathcal{D} \), and the harmonic function \( u(P) = \omega(P, E) \), then the harmonic measure of the set of points of \( E \), at which the nontangential boundary values of the function \( u \) differ from 1, is equal to zero.


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THEOREM. If \( u \) is a subharmonic function in a Lipschitz domain \( D \) and \( E = \{ Q \in \partial D: \lim_{P \to Q} u(P) = -\infty \} \), then \( \omega(P, E) = 0 \).

Proof. The boundary \( \partial D \) can be covered by a finite number of reduced standard cylindrical neighborhoods \( \partial D \subseteq \bigcup_{j=1}^{N} V(\bar{x}, \delta_{0}, Q_{j}) \). We denote

\[
V_{j} = V(r_{0}, \delta_{0}, Q_{j}), \quad \bar{V}_{j} = V(\bar{x}, \delta_{0}, Q_{j}), \quad E_{j} = E \cap \bar{V}_{j}, \quad A_{j} = \bar{V}_{j} \cap \partial D.
\]

It is sufficient to verify that \( \omega(P, E_{j}) = 0 \).

We fix \( j \) and we consider the corresponding system of coordinates \((x, y)\) for the point \( Q_{j} \). We select \( \rho = 2\epsilon \) and \( h > 0 \) so that for \( Q \in A_{j} \) the nontangential cone \( \Gamma(h, \rho, Q) \subset V_{j} \). On the set \( A_{j} \) we define a sequence of continuous functions

\[
g_{n}(x, y) = \sup \{ u(\xi, \eta): \rho |\xi - x| < \eta - y < \frac{h}{n}, \frac{h}{n} < \eta - y \}.
\]

The assertion \( g_{n}(x, y) \to -\infty \) is equivalent to the statement

\[
\lim_{(\xi, \eta) \to (x, y)} u(\xi, \eta) = -\infty.
\]

Therefore,

\[
E_{j} = \{ (x, y) \in A_{j}: \lim_{n \to +\infty} g_{n}(x, y) = -\infty \} = \bigcup_{Q \in E} \Gamma(h, \rho, (x, y))
\]

is a Borel set. We show that \( \omega(P, E_{j}) = 0 \). Otherwise, there exists a compactum \( B \subset E_{j} \) for which \( \omega(P, B) > 0 \). In the domain

\[
G = (D \cap V_{j}) \cap \bigcup_{Q \in E} \Gamma(\rho, \rho, Q)
\]

we define the harmonic function \( w(x, y) = \omega((x, y), B) \). We estimate \( w \) on \( \partial G \setminus B \). Part of this set, consisting of the points situated at a distance from \( B \) not smaller than \( r_{0}/4 \), form a compactum \( K \subset D \) with \( \max_{K} w = M_{1} < 1 \). If the point \((\xi, \eta) \in \partial G \setminus B \) is situated at a distance from \( B \) smaller than \( r_{0}/4 \), then \((\xi, \eta) \notin \bigcup_{Q \in E} \Gamma(\rho, \rho, Q) \) and \( \eta > \varphi(\xi) \). For the estimation we make use of Lemma 1. The point \( \xi \in \mathbb{R}^{n} \) is situated at a positive distance \( 2r \) from the projection of \( B \) onto \( \mathbb{R}^{n} \). \( \Delta((\xi, \varphi(\xi)), 2r) \cap B = \emptyset \). We denote by \( \tilde{x} \) one of the points of the projection of \( B \) onto \( \mathbb{R}^{n} \), situated at a distance \( 2r \) from \( \xi \). Then

\[
(\tilde{x}, \varphi(\tilde{x})) \in B, \quad (\xi, \eta) \notin \Gamma(\rho, \rho, (\tilde{x}, \varphi(\tilde{x})))
\]

and

\[
0 < \eta - \varphi(\xi) \leq (\eta - \varphi(\tilde{x})) + (\varphi(\xi) - \varphi(\tilde{x})) \leq (\rho + c)|\tilde{x} - \xi| = 6cr = sr < \delta_{0},
\]

since \( r < r_{0}/8 \) and \( cr_{0} < \delta_{0}/2 \). Our point

\[
(\xi, \eta) \in V(r, sr, (\tilde{x}, \varphi(\tilde{x}))).
\]

Therefore \( \omega((\xi, \eta), \Delta((\xi, \varphi(\xi)), 2r)) \geq a \) and \( w(\xi, \eta) \leq 1 - a = M_{2} < 1 \). We have assumed that \( \omega(P, B) > 0 \). By Lemma 2 there exists a point \((x_{0}, y_{0}) \in B \) at which \( w \) has nontangential boundary value equal to 1. We select \((x_{1}, y_{1}) \in G \) so close to \((x_{0}, y_{0}) \) that \( w(x_{1}, y_{1}) = M_{3} > \max\{M_{1}, M_{2}\} \) and, moreover, \( u(x_{1}, y_{1}) \neq -\infty \). We denote \( M_{0} = \sup_{G} u < +\infty \). For \( \alpha > 0 \) we define in \( G \) the subharmonic function

\[
\Omega_{\alpha}(x, y) = u(x, y) + \alpha(w(x, y) - w(x_{1}, y_{1})).
\]