PROJECTIONS ON $L^p$-SPACES OF POLYANALYTIC FUNCTIONS

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The fundamental result: for an arbitrary bounded, simply connected domain $\Omega$ in $\mathbb{C}$, the subspace $L_{n,m}^p(\Omega)$ of the space $L^p(\Omega, \sigma)$ ($\sigma$ is the plane Lebesgue measure, $p \geq 1$), consisting of the $(m, n)$-analytic functions in $\Omega$, is complemented in $L^p(\Omega, \sigma)$ (a function $f$ is said to be $(m, n)$-analytic if $(\partial^{m+n}/\partial \bar{z}^m \partial z^n)f = 0$ in $\Omega$). Consequently, by virtue of a theorem of J. Lindenstrauss and A. Pelczyński, the space $L_{n,m}^p(\Omega)$ is linearly homeomorphic to $l^p$. In particular, for $m = n = 1$ we obtain that the space of all harmonic $L^p$-functions in $\Omega$ is complemented in $L^p(\Omega, \sigma)$. This result has been known earlier only for smooth domains.

We shall consider a plane domain $\Omega$, the space $L^p(\Omega) (= L^p(\Omega, \sigma)$, where $\sigma$ is the Lebesgue plane measure), and its subspace $L_{n,m}^p(\Omega) (\Omega)$, consisting of all possible functions $f \in L^p(\Omega)$ satisfying the equation

$$\partial^n \bar{\partial}^m f = 0,$$

where $\partial = 1/2 (\partial/\partial x - i \partial/\partial y)$ and $\bar{\partial} = 1/2 (\partial/\partial x + i \partial/\partial y)$ (the derivatives are understood in the sense of the theory of distributions).

The functions satisfying equation (1) are called said to be $(m, n)$-analytic. All of them are real analytic. The space $L_{n,m}^p(\Omega)$ is a generalization of the known $L^p$-spaces of analytic, harmonic, and polyanalytic functions and coincides with them for definite indices $n$ and $m$. Regarding polyanalytic functions see the survey [1].

In this paper we investigate the problems of the linear homeomorphic classification of the spaces $L_{n,m}^p(\Omega)$.

In 1971, J. Lindenstrauss and A. Pelczyński [2] have discovered that for $p > 1$ the space $L_0^p(\mathbb{D}) = L_{0,1}^p(\mathbb{D})$ of analytic functions in the unit disc $\mathbb{D}$ is isomorphic to the space $l^p$ of sequences. They have also formulated the problem of finding analogous realizations in the form of spaces of sequences for other classes of functions, important in analysis. At the foundation of the proof of the isomorphism theorem one has the following

**Lemma** (J. Lindenstrauss and A. Pelczyński). Let $\Omega$ be a domain in $\mathbb{R}^n$, let $\mu$ be a finite positive Borel measure in $\mathbb{R}^n$, whose support coincides with the closure of the domain $\Omega$, let $X$ be an infinite-dimensional subspace in $L^p(\Omega, \mu)$, possessing the following properties:

1) $X$ is complemented in $L^p(\Omega, \mu)$;
2) the weak convergence of a sequence in $X$ implies the uniform convergence on compact subsets in $\Omega$.

Then $X$ is isomorphic to $l^p$.

For the $L^p$-spaces of solutions of elliptic equations, condition 2) of the lemma is satisfied (see, for example, [3]). Here the complementability question is crucial. The corresponding projection onto the space $L_{n,m}^p(\Omega)$ for $1 < p < \infty$, in a domain with a sufficient smooth boundary, can be obtained by extending "by continuity" the orthogonal projection from $L^2(\Omega)$ onto the space $L_{n,m}^p(\Omega)$ into $L^p(\Omega)$.

A. A. Solov'ev [4] and J. Burbea [10] have investigated geometric conditions on the boundary of the domain, under which the orthogonal projection onto the space $L_{0,1}^2(\Omega)$ of analytic functions and onto the space $L_{1,1}^2(\Omega)$ of harmonic functions is continuous in the $L^p$-metric. For a definite class of domains these conditions are necessary and sufficient for the continuity of the corresponding projection.
In this paper we construct the projection onto the space $L_{n,m}^p(\Omega)$ for any simply connected domain $\Omega$ in the plane. The linear homeomorphism $A^{n,m}$, used for this, of the spaces $L^p(\Omega)$ and $L^p(D)$, such that $A^{n,m}(L_{n,m}^p(\Omega)) = L_{n,m}^p(D)$, is the analogue of the linear isometry

$$A^{0,1} f = (\tau \circ f) \cdot (\tau')^{1/p} ,$$

(2)

preserving the classes of analytic functions in the disc $D$ and in the simply connected domain $\Omega$, generated by the conformal mapping $\tau: D \to \Omega$.

The desired projection onto $L_{n,m}^p(\Omega)$ for a simply connected domain $\Omega$ is obtained with the aid of a transplant (by means of the homeomorphism $A^{n,m}$) of the classical projection $L^p(D) \to L_{n,m}^p(D)$, where $D$ is the open unit disc (by the classical projection we mean that one which is orthogonal for $p = 2$).

An analogous construction goes through also for $p = 1$ but in this case, as the projection onto the space $L_{n,m}^1(D)$ of $(m, n)$-analytic functions in the unit disc, instead of the classical projection one selects another one, whose construction is connected with the use of weighted spaces with the weight $(1 - |z|^2)^{\alpha}$ (see, for example, [5]).

Required estimates of certain integrals are given in Sec. 1. The reproducing kernel for the space $L_{n,m}^p(D)$ is investigated in Sec. 2. In Sec. 3 we construct explicitly the isomorphism $A^{n,m}$. The proof of the continuity of the operator $A^{n,m}$ in the $L^p$-metric is given in Sec. 4.

By $C$ we denote a constant, whose value is inessential and it can vary in the course of the computations.

1. Estimates of Certain Integrals and of Derivatives of Univalent Functions

In W. Rudin's book [6], at the investigation of the complementability problems of spaces of analytic functions, one derives the following estimate.

1.1. LEMMA [6]. Let $\alpha > -1, \beta > 0$; there exists a constant $c = c(\alpha, \beta)$ such that

$$\int_D \frac{(1 - |w|^2)^{\alpha}}{|1 - \overline{z}w|^2} \, dE(w) \leq \frac{c}{(1 - |z|^2)^{\beta}} .$$

(3)

We prove an analogous estimate.

1.2. LEMMA. Let $\alpha > -1, \beta > 0, \gamma > -2$; there exists a constant $c = c(\alpha, \beta, \gamma)$ such that

$$\int_D \frac{|z - w|^\gamma (1 - |w|^2)^{\alpha}}{|1 - \overline{z}w|^{2 + \alpha + \beta}} \, dE(w) \leq \frac{c}{(1 - |z|^2)^{\beta}} .$$

(4)

Proof. We mention at once that one has to prove the estimate (4) only for $z$ close to the boundary of the circle.

If $\gamma > 0$, then, making use of the inequality $|z - w| \leq |1 - z\overline{w}|$, we reduce the estimate (4) to the estimate (3).

Let $-2 < \gamma < 0$. We partition the circle into two sets:

$$A_z = \{ w : 1 - |z|^2 < \frac{4}{\gamma} |1 - z\overline{w}| \}$$

and

$$B_z = \{ w : 1 - |z|^2 \geq \frac{4}{\gamma} |1 - z\overline{w}| \} .$$

On the set $A_z$ we have the inequality

$$|z - w| \geq |\overline{z}(z - w)| = |(1 - w\overline{z}) - (1 - |z|^2)| \geq \frac{4}{\gamma} |1 - z\overline{w}| ,$$

and, therefore,