FATOU'S THEOREM ON NONTANGENTIAL LIMITS AND PROBLEMS OF EXTENSION TO AN IDEAL BOUNDARY

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An affirmative answer is given to a question of S. Axler and A. Shields regarding the possibility of the continuous extension to the Shilov boundary $M(L^\infty)$ of the algebra $H^\infty$ of arbitrary continuous (not necessarily bounded) functions in the unit circle, having almost everywhere nontangential limits. At the same time a description is obtained, in terms of the continuous extension to part of the boundary $M(H^\infty)$, of the maximal class of continuous functions for which Fatou's theorem on nontangential limits holds.

1. Introduction

In this paper we shall find the maximal class of continuous (not necessarily analytic or harmonic) functions, for which Fatou's theorem on nontangential limits holds. It turns out that this boundary property is equivalent to the existence of a continuous extension to the Shilov boundary $M(L^\infty)$ of the algebra $H^\infty$ of bounded analytic functions. The given result sheds light on the nature of nontangential limits, used everywhere in analysis, and establishes a close relationship between Fatou’s theorem on nontangential limits and the topology of the space of maximal ideals $M(H^\infty)$ of the algebra $H^\infty$ [1].

Apparently, the problem of the nontrivial description of the algebra $H_{L^\infty}$ of all functions, continuous and bounded in the open circle and for which Fatou’s theorem holds, has been mentioned for the first time in [2], where one has found the space of maximal ideals of the algebra $H_{H^\infty}$ of all functions, continuous and bounded in the open circle and having analytic boundary values. In particular, in [2, 3] one has remarked that the space of maximal ideals $M(H_{L^\infty})$ is the Shilov boundary of the algebra $H_{H^\infty}$.

In preprint [4] one has considered the set $\mathcal{H}$ of all complex-valued functions, continuous in the circle $\mathcal{H}$ (not necessarily bounded), having almost everywhere angular limits (possibly also $\infty$). It will be convenient to consider such functions as continuous mappings of the circle into the compactum $\overline{C} = C \cup \{\infty\}$, i.e., $f : \mathcal{H} \to \overline{C}$, where $\mathcal{H} = \{z : |z| < 1\}$ is the open circle.

By Carleson’s “corona” theorem, the space of the maximal ideals $M(H^\infty)$ of the uniform algebra $H^\infty$ of all bounded analytic functions in the unit circle $\mathcal{H}$ contains the open circle $\mathcal{H}$ as an everywhere dense subset [1]. The compactum $M(L^\infty) \subseteq M(H^\infty) \setminus \mathcal{H}$ is the Shilov boundary of the algebra $H^\infty$. By $\mathcal{H} \cup M(L^\infty)$ we shall mean the subspace $M(H^\infty)$ with the induced topology.

**THEOREM A** (Axler–Shields, [4]). Let $f$ be a continuous function in the open circle $\mathcal{H}$ (not necessarily bounded), admitting a continuous extension to the Shilov boundary $M(L^\infty)$:

$$\tilde{f} : \mathcal{H} \cup M(L^\infty) \to \overline{C}.$$ 

Then $f \in \mathcal{H}$.

In the preprint [4] one has posed the following question: is there a converse to Theorem A, i.e., if $f \in \mathcal{H}$, does the function $f$ have a continuous extension to $M(L^\infty)$,

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The fundamental result of this paper is a positive answer to this question by S. Axler and A. Shields.

The proof splits in a natural manner into two steps: the case of bounded functions (Sec. 3) and that of unbounded functions (Sec. 5). In the brief Sec. 4 the theorem is proved for functions with boundary values from BMO. However, this result will not be used in the sequel. Section 2 is devoted to T. Gamelin’s refined theorem, on which our proof is based. In Remark 2 we give an example of a function from the Bloch class, having no continuous extension to the set of irregular points of $M(H^\infty)$, lying outside the Shilov boundary $M(L^\infty)$.

Another proof of Theorem 1 (the case of bounded functions) has been kindly communicated to me by C. Bishop in June 1989.

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2. Preliminary Information

By a Stolz angle of opening $\beta$ we mean the open triangle in a circle with vertex at the point $z_0 \in \mathbb{T} = \{z: |z| = 1\}$

$$\Gamma_{z_0}(\beta) = \{ z \in \mathbb{D} : \frac{|z_0 - z|}{1 - |z|} < \frac{1}{\cos \frac{\beta}{2}} \} .$$

Our main tool will be T. Gamelin’s little-known theorem [8, p. 31] (see below) on the description of the growth $[S]_{M(H^\infty)} S$ in the topological space $M(H^\infty)$ of an arbitrary set $S \subseteq \mathbb{T}$, where $[S]_{M(H^\infty)}$ is the closure of the set $S$ in the space $M(H^\infty)$. The standard results on the space of maximal ideals of $M(H^\infty)$ and the Shilov boundary $M(L^\infty)$ of the algebra $H^\infty$ are collected in [1].

Definition 1. Let $S$ be a subset of the open disc $\mathbb{D}$ and let $z_0 \in \mathbb{T}$ be a limit point (in the Euclidean topology) of $S$. The point $z_0$ is said to be a nontangential limit point of $S$ if there exists a Stolz angle $\Gamma_{z_0}(\beta)$, containing, starting with some index, a sequence $\{z_n\}$ of points of $S$ such that $\lim_{n \to \infty} |z_n - z_0| = 0$.

The set of all nontangential limit points of $S$ will be denoted by $F(S)$.

We need also the concept of the lifting of a Borel set $E \subseteq \mathbb{T}$ to $M(L^\infty)$. Let $E$ be a Borel set of positive Lebesgue measure; then the characteristic function $\chi_E \neq 0$. Its Gelfand transform $\hat{\chi}_E$ is idempotent in $L^\infty$ and, therefore, it is the characteristic function of some open-closed set from $M(L^\infty)$, which will be denoted by $\hat{E}$.

Definition 2. An open-closed set $\hat{E} \subseteq M(L^\infty)$ for which $\hat{\chi}_E = \chi_E$ is called a lifting of the set $E \subseteq \mathbb{T}$.

We note that the lifting $E \rightarrow \hat{E}$ establishes a one-to-one correspondence between the Borel subsets, modulo sets of Lebesgue measure zero, and the open-closed subsets of $M(L^\infty)$ [11, p. 32 of the Russian edition].

Now we give T. Gamelin’s theorem.

THEOREM B ([8, p. 31]). Let $S$ be an arbitrary subset of the circle $\mathbb{D}$. Then

$$[S]_{M(H^\infty)} \cap M(L^\infty) = \hat{F}(S) .$$

COROLLARY B. Under the conditions of Theorem B we have

$$[S]_{M(H^\infty)} \cap M(L^\infty) \neq \emptyset \iff m(\hat{F}(S)) > 0 ,$$

where $m$ is the Lebesgue measure on $\mathbb{T}$.

Example. Let $S$ be an arbitrary subset of the circle, having a finite number of limit points (in the Euclidean topology) on the circumference $\mathbb{T}$. Then $[S]_{M(H^\infty)} \cap M(L^\infty)$. 

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