EXTREMAL PROBLEMS IN THE CLASS $\Sigma(\tau)$

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Let $L_f(r) = \{ w = f(z), |z| = r \}, 1 < r < \infty$, be a level line of the function $f(z) \in \Sigma$. Sharp upper bounds are obtained for the diameter of the curve $L_f(r)$ in the class $\Sigma(\tau)$ for functions $f(z) = z + \alpha_0 + \alpha_1 z^{-1} + \ldots \in \Sigma$ for which there exists a domain $\Delta_f$ that complements the exterior of the unit disk and has conformal radius at the point $w = 0$ satisfying the condition $R(\Delta_f, 0) \geq \tau, 0 < \tau < 1$. Also, a set of values is found for the coefficient $\alpha_1$ in the class $\Sigma(\tau)$.

Let $\Sigma$ be the class of univalent functions $w = f(z)$ that are meromorphic in $U^* = \{ z: |z| > 1 \}$ and, in the neighborhood of the point $z = \infty$, have the expansion

$$f(z) = z + \alpha_0 + \alpha_1 z^{-1} + \ldots \quad (1)$$

We use $\Sigma(\tau), 0 < \tau < 1$, to denote the class of functions $f(z) \in \Sigma$ for which the set $\mathbb{C} \setminus f(U^*)$ contains a domain $D, 0 \in D$, satisfying the condition

$$R(D, 0) \geq \tau,$$

where $R(D, 0)$ is the conformal radius of the domain $D$ relative to the point $z = 0$. The class $\Sigma(\tau)$ was considered by Jenkins in [1], where a set of values for the coefficient $\alpha_0$ in the class was found.

For $f \in \Sigma$, let $L_f(\tau) = \{ w: |f^{-1}(w)| = \tau \}, 1 < \tau < \infty$, be a level line of the function $z = f^{-1}(w)$ inverse to $w = f(z)$. In the present paper we solve the following two extremal problems in the class $\Sigma(\tau)$: Theorem 1 provides a sharp bound for the diameter of the level lines $L_f(\tau)$, and in Theorem 2 we find a set of values for the coefficient $\alpha_1$ in the class $\Sigma(\tau)$. The proofs of these theorems are based on investigation of certain problems on extremal partition of $C$ that are equivalent to the modulus problem for pairs of homotopic classes of curves (see [3]).

In §1 of the present paper we prove auxiliary propositions that are needed later. In §2 we consider a geometric description of extremal pairs of domains and investigate the properties of functions of these pairs in the problems on extremal partition under discussion. In §3 we solve the indicated extremal problems in the class $\Sigma(\tau)$.

Henceforth, $m(G)$ denotes the modulus of a doubly connected region $G$ with respect to the family of curves separating its boundary components; $m(D, z_0)$ is the reduced modulus of a singly connected domain with respect to a point $z_0 \in D$;

$$U = \{ z: |z| < 1 \}, \quad C_l(a) = \{ z: |z - a| = \tau \}, \quad C_\infty = C_\infty(3), \quad R_+ = \{ z: z > 0 \}.$$
Let $B_k, 0 \in B_k, k = 1, 2$, be singly connected domains that are symmetric relative to the real axis $\mathbb{R}$ and have boundaries that intersect in only two points, $a$ and $\bar{a}$; let $G_k = B_k \setminus E$, where $E$ is a continuum that is symmetric with respect to $\mathbb{R}, 0 \in E \subset B_1 \cap B_2$.

**Lemma 1.** Let $(B_1 \cap \mathbb{R}^+) \supset (B_2 \cap \mathbb{R}^+)$, let $f_1(z)$ be a conformal homeomorphism of $B_2$ onto $B_1$, $f_1(0) = 0, f_1'(0) = 1$, let $m(G_1) = m(G_2)$, let $f_2(z)$ be a conformal homeomorphism of $G_2$ onto $G_1$, $f_2(\partial E) = \partial E$, and let $f_2(z) > 0$ for $z > 0, z \in G_2$. Then

\[
\frac{f_1'(z)}{f_1'(z)} > q, \quad \text{for } t > 0, \ t \in B_2, \tag{3}
\]

\[
\frac{f_2'(z)}{f_2'(z)} > q, \quad \text{for } t > 0, \ t \in G_2. \tag{4}
\]

**Proof.** We will prove inequality (4), since inequality (3) can be proved similarly. Let $\omega_k(z)$ be a harmonic measure of the set $\partial E$ relative to the domain $G_k, k = 1, 2$. We consider the function

\[
h(z) = \omega_1(z) - \omega_2(z).
\]

which is harmonic in $G = G_1 \cap G_2$. Then $a$ and $\bar{a}$ divide the boundary of $G$ into arcs $\Gamma_1$ and $\Gamma_2$, $\Gamma_k \subset \partial G_k$, and here $h(z) = 0$ if $z \in \partial E$, $h(z) > 0$ if $z \in \Gamma_2$, and $h(z) < 0$ if $z \in \Gamma_1$.

Let $\gamma = \{z \in \Gamma_1 \cap \Gamma_2 \mid h(z) = 0\}$; then $a, \bar{a} \in \gamma$. It follows from the properties of level lines of harmonic functions and the symmetry of $G$ that $\gamma$ either consists of two curves $\gamma'$ and $\gamma''$ that are symmetric with respect to $\mathbb{R}$, where $\gamma'$ belongs to the upper half-plane and connects the point $a$ and the continuum $E$, or it is a curve in $G$ that is symmetric with respect to $\mathbb{R}$ and connects the points $a$ and $\bar{a}$. We will show the second case is impossible. The set $G \setminus \gamma$ consists of two components $G'$ and $G''$, where $\partial G' \supset \Gamma_2$. If $\gamma \cap E \neq \emptyset$, one of these components, say, $G'$, is doubly connected and contains $\partial E$ without it being the boundary. Then, by the principle of the maximum for harmonic functions, $\omega_1(z) > \omega_2(z)$ if $z \in G'$.

Let $\gamma(\lambda) = \{z \mid \omega_k(z) = \lambda, k = 1, 2; 0 < \lambda < 1\}$. It is known that $\gamma_1(\lambda) = f_2(\gamma_2(\lambda))$. It follows from the indicated inequality that for values of $\lambda$ sufficiently close to 1, the level line $\gamma_2(\lambda)$ separates $\gamma_1(\lambda)$ from the continuum $E$, which contradicts the conformal invariance of the modulus of a doubly connected domain. As a result, both components $G'$ and $G''$ are singly connected, and $\omega_1(z) > \omega_2(z)$ for $z \in G'$. Since $\omega_2(z) = \omega_1(f_2(z))$, for $r > 0, r \in G_2$, we have $\omega_1(r) > \omega_1 f_2(r)$. Inequality (4) now follows.

**Remark 1.** A proposition similar to Lemma 1 holds for pairs of symmetric domains whose boundaries intersect in four points. In addition, we can weaken the conditions of the Lemma by permitting sections of the boundaries of the domains under consideration to overlap.

**1.2. Lemma 2.** Let $D$ be a singly connected domain of hyperbolic type, and let $t$ be a point of $D$. Then $m(D,t)$ is a real analytic superharmonic function of $u$ and $v$, where $t = u + iv$.

Indeed, let $\xi = f(z)$ be a fixed conformal homeomorphism of $D$ onto $U$. Then

\[
m(D,\xi) = -\frac{1}{2\pi} \log \left| \frac{\xi'(t)}{1 - |\xi'(t)|^2} \right|.
\]

Since $f'(t) \neq 0$, $\log | f'(t) |$ is a harmonic function in $D$. The second term on the right side of (5) is a superharmonic function, as the superposition of analytic and superharmonic functions (see, for example, [3]).

**Lemma 3.** Let $D$ be a singly connected domain that is circularly symmetric with respect to the ray $\mathbb{R}_+$, and let $(x_0, 0) \subset D$, where $x_0 < 0$. Then there exists an $x_1$ such that the function $m(D,x)$ increases for $x_0 \leq x \leq x_1$, and here $x_1 = 0$ if $D$ is a disk with center at the point $z = 0$, and $x_1 > 0$ otherwise.

**Proof.** Let $x_0 \leq x' < x'' \leq 0$. It follows from Lemma 2 and the principle of the minimum for superharmonic functions that

\[
m(D, x'') > \min_{0 \leq \theta < 2\pi} m(D, |x'| e^{i\theta}) = m(D, x').
\]

This last equation was proved for circularly symmetric domains in [4] (see also [5]).

Let $[x_0, x_J], x_0 < x_1$, be the largest interval beginning at the point $x_0$ in which the function $m(D,x)$ increases. If $x_1 = 0$, then $z = 0$ is a point of local maximum for the function $m(D,z)$. Let