THE $\varepsilon$-INCLUSION CONDITION FOR N-COMPACTA

V. A. Shlyk

Criteria for NED-sets and N-compacta generating normal domains are given in terms of $\varepsilon$-inclusion.

1. It is known [1, Theorem 10] that if $E$ is an NED-set in the complex plane $C$, or, in other words, a removal set for
the class of regular functions with bounded Dirichlet integral, the two-dimensional Lebesgue measure $L_2(E)$ of the set $E$ is equal
to zero, and any two points $a$ and $b$ of $C\setminus E$ can be connected by a curve $\gamma \subset C\setminus E$ of length $s(\gamma)$ arbitrarily close to the distance
d$(a,b)$ between them (the permeability property of the compactum $E$).

A weaker permeability relative to the real axis is characteristic of an N-compactum $E$ whose complement to $\mathbb{C} = C$
$\cup \{\infty\}$ is a minimal domain in the sense of Koebe [2, 3] (or, in other words, a normal domain in the sense of Grotzch [2,
3]) under a mapping onto the plane with linear cuts parallel to the real axis. Namely, $L_2(E) = 0$ and any two points $a, b \in C\setminus E, \Re a = \Re b$, may be connected by a curve $\gamma \in C\setminus E$ whose length is arbitrarily close to the distance $d(a,b)$. On the other
hand, a classical example of Koebe [4] shows that permeability of a compactum $E$ with respect to all direction does not
guarantee minimality of the domain $\mathbb{C} \setminus E$ in the above sense.

Below we introduce the notion of $\varepsilon$-inclusion of a compactum with respect to a line. It generalizes the corresponding
property of permeability and makes it possible to establish new exact characteristics for N-compacta and NED-sets.

2. Let $H_k$ and $L_k$ respectively be $k$-dimensional Hausdorff and Lebesgue measures. We set $L(C) = \{ z \in \mathbb{C} : \Im z = y \}$. We will say that a compactum $E \subset C$ satisfies the $\varepsilon$-inclusion condition with respect to the real axis if:

1) $L_2(E) = 0$;

2) for each Borel function $\rho : C \to [0, \infty], \rho \in L^2(C)$, we can, for an arbitrary given $\varepsilon > 0$, find, on the line $l(y)$, $l(y) \cap E \neq \emptyset$, a finite number of pairwise disjoint segments $[a_m, b_m]$, $m = 1, m$, that cover the compactum $l(y) \cap E$ and have
total length less than $H_1(l(y) \cap E) + \varepsilon$. In this case, for $L_1$-almost all such $y$, the ends of the segment $[a_m, b_m]$ do not belong
to $E$ and are connected in $C\setminus E$ by a rectifiable curve $S_m$ so that

$$\sum_{m=1}^{m} \int_{S_m} \rho(x) \, ds = \varepsilon. \tag{1}$$

Similarly, we introduce the notion of $\varepsilon$-inclusion of a compactum $E$ with respect to a line $l_\theta = l_{\theta + \pi}$ forming an angle$
\theta$ with the real axis. Moreover, if we substitute $L^p(\mathbb{R}^n)$ for $L^2(C)$ and $L_{n-1}$ for $L_1$ in the definition of $\varepsilon$-inclusion, we can
analogously state the condition for $\varepsilon$-inclusion for a compactum $E \subset \mathbb{R}^n, n \geq 2$, with respect to, say, the $x_j$ coordinate axis.

3. Let $\Pi = \{ Z = x + iy \in \mathbb{C} : a_1 < x < b_1, a_2 < y < b_2 \}$ be a nondegenerate coordinate rectangle. We set

$$A_1 = (a_1, a_2), \quad A_2 = (b_1, a_2), \quad A_3 = (b_1, b_2), \quad A_4 = (a_1, b_2),$$

$$\sigma_0 = \{ z \in \mathbb{C} : x = a_1, a_2 < y < b_2 \}, \quad \sigma_1 = \{ z \in \mathbb{C} : x = b_1, a_2 < y < b_2 \}.$$

Let $\Gamma(\sigma_0, \sigma_1, \Pi \setminus E)$ (resp. $\Gamma(\sigma_0, \sigma_1, \Pi)$) be the family of all rectifiable curves located in $\Pi \setminus E$ (resp. in $\Pi$) that connect $\sigma_0$ and $\sigma_1$.

We denote its $2$-modulus [3] by $m_2(\sigma_0, \sigma_1, \Pi \setminus E)$ (resp. by $m_2(\sigma_0, \sigma_1, \Pi)$).

**THEOREM 1.** $E$ is an N-compactum if and only if $E$ satisfies the $\varepsilon$-inclusion condition with respect to $l_0$.

**Proof.** Necessity. Let $\Pi \supset \Pi$. Then (see [3]) $m_2(\sigma_0, \sigma_1, \Pi \setminus E) = m_2(\sigma_0, \sigma_1, \Pi \setminus E), G = \Pi \setminus E = 0$. Let $G_j$ be a finitely connected subdomain of $G$ whose outer boundary coincides with the outer boundary of $G$, where

$$\Pi_j = A_1 A_2 A_3 A_4 \setminus A_j.$$

It is known [4] that there exists a univalent conformal mapping $f_j$ of $G_j$ onto a domain $B_j$ whose outer

Translated from Zapiski Nauchnykh Seminarov Leningradskogo Otdeleniya Matematicheskogo Instituta im. V. A.
boundary is the contour of the rectangle \( \Pi_j + A_1A_2A_3A_4 \), and whose inner boundaries are points or line segments parallel to the real axis. Here \( \Pi_j \subset \Pi \) and

\[
\begin{align*}
\tilde{f}_j(A_1) &= A_1, \\
\tilde{f}_j(A_2) &= A_2, \\
\tilde{f}_j(A_3) &= A_3, \\
\tilde{f}_j(A_4) &= A_4.
\end{align*}
\]

In virtue of the necessity of the condition of the theorem, we can assume that the sequence of \( \tilde{f}_j, j \in \mathbb{N} \), converges uniformly inside \( G \) to the function \( f(z) = z \). We set \( E^j = \Pi_j \setminus B_j \). We denote the function inverse to \( w = \tilde{f}_j(z), z \in G_j \), by \( \tilde{g}_j(w) \), where \( w \in B_j \), and let \( \rho_j(w) = \rho_j(g_j(w)) | g_j'(w) | \) for \( w \in B_j \) and \( \rho_j(w) = g_j'(w) = 0 \) for \( w \in \Pi \setminus B_j \). Since \( g_j \) and \( g_j' \) converge as \( j \to \infty \) uniformly inside \( G \) to the identity mapping and unity, respectively, it follows from Luzin's theorem that \( \rho_j \) converges pointwise \( L^2 \)-almost everywhere in \( \Pi \) to \( \rho \). Similarly, in view of the condition \( \rho \in L^2(\Pi) \) and the univalence of \( \tilde{f}_j \), for a given \( \delta_1 > 0 \), the compactum \( E \) must belong to an open set \( \varnothing \subset \Pi \) such that for \( j \geq j_0 \) we have

\[
\int_0 \rho^2 \, dL_2 < \delta_1, \quad \int_0 \rho^2 \, dL_2 < \delta_1.
\]

Then, by Egorov's theorem, we have

\[
\lim_{j \to \infty} \int_{\Pi} |\rho(z) - \rho_j(z)|^2 \, dL_2 = 0.
\]

It follows that

\[
\lim_{j \to \infty} \int_{E(j)} \rho(x,y) - \rho_j(x,y)^2 \, dx = 0
\]

for all \( y \in (a_2,b_2) \setminus K_1 \), where \( K_1 \subset (a_1,b_1) \) and \( H_1(K_1) = 0 \).

Let \( K_2 \) be the set of all points \( y \in (a_2,b_2) \) for which at least one of the following conditions is satisfied:

1) \( \int_{E(j)} \rho(x,y) \, dx = \infty \);

2) \( l(y) \cap \bigcup_{j=1}^\infty E_j \neq \varnothing \);

3) \( H^1(l(y) \cap E) \geq 0 \);

4) the function \( \int_{l(y)} \rho(x,y) \, dx, y \in (a_2,b_2) \), is not approximatively continuous at the point \( y \).

By construction \( H^1(K_2) = 0 \). We select \( e_1 > 0 \) and choose some point \( y_0 \in (a_2,b_2) \setminus (K_1 \cup K_2) \). We cover \( E \cap l(y_0) \) with a finite number of linear disjoint segments \( q_1, \ldots, q_k \subset l(y_0) \) whose total length is less than \( e_1 \) and whose ends are located within \( E \). From the ends of these segments, we draw codirected linear segments \( m_1, \ldots, m_s \subset \Pi \setminus E \) of the same length parallel to the \( y \)-axis. Shortening the segments \( q_1, \ldots, q_k \) if necessary, we subject the set of segments \( m_1, \ldots, m_s \) to the condition

\[
\sum_{p=1}^s \int_{m_p} \rho \, dH^1 < e_1,
\]

where \( e_1 \) is a given positive number.

We choose \( y_1 \in (a_2,b_2) \setminus (K_1 \cup K_2) \) sufficiently close to \( y_0 \) and such that the line \( l(y_1) \) intersects all of the segments \( m_1, \ldots, m_s \), forming, in this case, pairwise disjoint segments \( \tilde{q}_1, \ldots, \tilde{q}_k \), where \( \{\text{Re } z: z \in \tilde{q}_p\} = \{\text{Re } z: z \in \tilde{q}_p\} \) for \( p = 1, \ldots, k \). We extend the segments \( \tilde{q}_1, \ldots, \tilde{q}_k \) somewhat to pairwise disjoint segments \( n_1, \ldots, n_k \) of \( l(y_1) \cap \Pi \) for which the arcs \( \tilde{q}_j(n_1), \ldots, \tilde{q}_j(n_k) \), as before, intersect the corresponding segments \( m_1, \ldots, m_s \) for \( j \geq j_0 \) and \( \sum_{p=1}^k H^1(n_p) < 2 \varepsilon_1 \). This can be done because of the uniform convergence of the functions \( f_j \) and \( g_j \) as \( j \to \infty \) to the identity mapping in a sufficiently small neighborhood of each segment \( \tilde{m}_j, \tilde{n}_j = \tilde{m}_j \). It follows from the Cauchy—Bunyakovski inequality and the choice of \( y_0 \) and \( y_1 \) that we have