Mixing and some integro-functional equations

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Summary. It is proved that the operator $P: L^1(0, \infty) \rightarrow L^1(0, \infty)$, given by $P(g) = \int_0^{\infty} \frac{g(x)}{cx} \, dx$, is completely mixing, i.e., $\|P^n g\|_1 \rightarrow 0$ for $g \in L^1(0, \infty)$ with $\int g \, dx = 0$. This implies that, for $c \in (0, 1)$, each continuous and bounded solution of the equation $f(x) = \int_0^{\infty} f(t) \, dt/(cx)$ ($x \in (0, 1]$) is constant.

1. Introduction

At the Twenty-sixth International Symposium on Functional Equations W. Walter ([6], Problem 32) put the following question.

Let $\varphi$ be a bounded continuous function on $(0, 1]$ and let $c \in (0, 1)$ be fixed. Does

$$\varphi(x) = \frac{1}{cx} \int_0^{cx} \varphi(t) \, dt \quad \text{for all } x \in (0, 1]$$

(1.1)

imply that $\varphi$ is constant?

Walter suggested the answer is yes and was able to prove it for $c \in [2^{-3/2}, 1]$. It is easy to observe that every bounded continuous solution of (1.1) on the interval $(0, 1]$ can be extended to the whole interval $(0, \infty)$ and this solution will be bounded and continuous. We show that for every $c > 0$ the only bounded continuous solutions of (1.1) on the interval $(0, \infty)$ are constants.

In the proof we use methods of ergodic theory and we show that the Markov operator corresponding to Eq. (1.1) is, in fact, completely mixing [2], which implies that, if $\varphi$ satisfies (1.1), then $\varphi$ is constant.

AMS (1980) Subject Classification: Primary 45A05, Secondary 45A35.

Manuscript received May 29, 1989, and in final form, February 8, 1990.
2. The main result

Denote by $L^1$ the space $L^1(0, \infty)$ with Lebesgue measure and by $L^\infty$ the conjugate space $L^\infty(0, \infty)$. By $D$ we denote the set of all densities $D = \{ f \in L^1 : f \geq 0, \int f = 1 \}$. We introduce two linear operators, $T : L^\infty \to L^\infty$ and $P : L^1 \to L^1$ given by

$$(T\varphi)(x) = \frac{1}{cx} \int_0^{cx} \varphi(t) \, dt$$

and

$$(Pg)(z) = \int_{z/c}^\infty \frac{g(x)}{cx} \, dx,$$

respectively. It is easy to check that $P$ is a Markov operator, i.e., $P$ is linear and $P(D) \subseteq D$. Moreover, the operator $T$ is conjugate to $P$, i.e., for every $\varphi \in L^\infty$ and $g \in L^1$

$$\int (T\varphi)g = \int (Pg)\varphi.$$

The operator $P$ is called completely mixing if

$$\lim \| P^n f - P^n g \|_1 = 0 \quad \text{for } f, g \in D. \quad (2.1)$$

Condition (2.1) is equivalent to the following:

$$\lim \| P^n g \|_1 = 0 \quad \text{for } g \in L^1 \text{ with } \int g = 0. \quad (2.2)$$

The property "completely mixing" is very strong, in particular it implies that the only fixed points of the conjugate operator are constants. Indeed, suppose that $P$ is completely mixing and $\varphi \in L^\infty$, $T\varphi = \varphi$. For every $g \in L^1$ with $\int g = 0$ we have

$$\int g\varphi = \int (T^n\varphi)g = \int (P^n g)\varphi \to 0 \quad \text{as } n \to \infty.$$

This implies that $\int g\varphi = 0$ provided $\int g = 0$, which means that $\varphi$ is constant.