A survey of Sylvester's problem and its generalizations

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Summary. Let \( P \) be a finite set of three or more noncollinear points in the plane. A line which contains two or more points of \( P \) is called a connecting line (determined by \( P \)), and we call a connecting line ordinary if it contains precisely two points of \( P \). Almost a century ago, Sylvester posed the disarmingly simple question: Must every set \( P \) determine at least one ordinary line? No solution was offered at that time and the problem seemed to have been forgotten. Forty years later it was independently rediscovered by Erdős, and solved by Gallai. In 1943 Erdős proposed the problem in the American Mathematical Monthly, still unaware that it had been asked fifty years earlier, and the following year Gallai's solution appeared in print. Since then there has appeared a substantial literature on the problem and its generalizations.

In this survey we review, in the first two sections, Sylvester's problem and its generalization to higher dimension. Then we gather results about the connecting lines, that is, the lines containing two or more of the points. Following this we look at the generalization to finite collections of sets of points. Finally, the points will be colored and the search will be for monochromatic connecting lines.

1. Introduction

Let a finite set of points in the plane have the property that the line through any two of them passes through a third point of the set. Must all the points lie on one line? Almost a century ago Sylvester (1893) posed this disarmingly simple question. No solution was offered at that time and the problem seemed to have been forgotten. Forty years later it resurfaced as a conjecture by Erdős: If a finite set of points in the plane are not all on one line then there is a line through exactly two of the points. In a recent reminiscence Erdős (1982, p. 208) wrote: "I expected it to be easy but to my great surprise and disappointment I could not find a proof. I told this problem to Gallai who very soon found an ingenious proof." In 1943 Erdős proposed the problem in the American Mathematical Monthly (Erdős 1943), still unaware that it had been asked fifty years earlier, and the following year Gallai's solution
appeared in print (Gallai 1944). Since then there has appeared a substantial literature on the problem and its generalizations, and two long-standing conjectures have recently been settled. But there are still unanswered questions, unsettled conjectures, and a survey paper at this time seems appropriate.

In the following two sections we review Sylvester's problem and its generalization to higher dimension. Then we will gather the properties of the connecting lines, that is, the lines containing two or more of the points. Following this we will look at the generalization to finite collections of sets of points. Finally, the points will be colored and the search will be for monochromatic connecting lines.

2. Sylvester's problem

The answer to Sylvester's question is negative in the complex projective plane and in some finite geometries (Coxeter 1948) so we restrict our attention to the ordinary real plane—Euclidean, affine, projective. (For interesting remarks and theorems in the "complex" case see L. M. Kelly (1986) and Boros, Füredi and Kelly (1989).) Let $P$ be a finite set of three or more noncollinear points in the plane. A line which contains two or more points of $P$ is called a connecting line (determined by $P$), and we call a connecting line ordinary if it contains precisely two points of $P$. In this terminology, Sylvester's theorem is: *Every set $P$ determines at least one ordinary line.* Because Gallai's proof came first we give it here although it played no role in subsequent developments. Here then is the affine proof by Gallai (1944) (also in: de Bruijn and Erdös 1948; Hadwiger, Debrunner and Klee 1964, p. 57; Croft 1967). Choose any point $p_1 \in P$. If $p_1$ lies on an ordinary line we are done, so we may assume that $p_1$ lies on no ordinary line. Project $p_1$ to infinity and consider the set of connecting lines containing $p_1$. These lines are all parallel to each other, and each contains $p_1$ and at least two other points of $P$. Any connecting line not through $p_1$ forms an angle with the parallel lines; let $s$ be a connecting line (not through $p_1$) which forms the smallest such angle (Figure 1a). Then $s$ must be ordinary! For suppose $s$ were to contain three (or more) points of $P$, say $p_2, p_3, p_4$ named so that $p_3$ is between $p_2$ and $p_4$ (Figure 1b). The connecting line through $p_3$...