One investigates the problem of the existence of an attractor $\mathcal{A}$ of the semigroup $S_t$, generated by the solutions of the nonlinear nonstationary equations

$$\frac{\partial u}{\partial t} = A(u), \quad u|_{t=0} = u_0(x); \quad S_t u_0 = u(t).$$

One proves a very general theorem on the existence of an attractor $\mathcal{A}$ of the semigroup $S_t$ for $t \to \infty$. One gives examples of differential equations having attractors: a second-order quasilinear parabolic equation, a two-dimensional Navier--Stokes system, a monotone parabolic equation of any order. One proves a theorem on the finiteness of the Hausdorff dimension of the attractor $\mathcal{A}$. One gives an estimate for the Hausdorff dimension of the attractor $\mathcal{A}$ for a two-dimensional Navier--Stokes system.

We investigate the attractors of nonlinear, nonstationary equations of the form

$$\frac{\partial u}{\partial t} = A(u), \quad u|_{t=0} = u_0(x). \tag{1}$$

We assume that problem (1) has a unique solution $u(t) \in E$ if $u_0 \in E$; $E$ is a given Banach space. The correspondence $u_0 = u(0) \mapsto u(t)$ where $u(t)$ is the solution of problem (1), defines a semigroup $S_t : E \to E, S_t u(0) = u(t)$.

We investigate the problem of the existence of an attractor $\mathcal{A}$ of the semigroup $S_t$ or, which is the same, of solutions of equation (1) for $t \to +\infty$. In addition, for a series of equations we give an estimate of the Hausdorff dimension of the attractor $\mathcal{A}$.

Definition 1 (see [1]). Let $S_t : E \to E, t > 0$, be a semigroup of operators in the Banach space $E$. A set $\mathcal{A} \subset E$ is said to be a maximal attractor of the semigroup $S_t$ if: a) for any bounded set $B \subset E$ and for any $\varepsilon > 0$ there exists $T = T(\varepsilon, B)$ such that for $t > T$ the set $S_t(B)$ is contained in the $\varepsilon$-neighborhood of the set $\mathcal{A}$; b) the set $\mathcal{A}$ is invariant relative to $S_t : \mathcal{A} \mapsto \mathcal{A}$.

In Sec. 1 we give a very general theorem regarding the existence of an attractor $\mathcal{A}$ of the semigroup $S_t$ for $t \to +\infty$. We give some examples of differential equations having attractors, namely a quasilinear second-order parabolic equation, the two-dimensional Navier--Stokes system, a monotone parabolic equation of any order. Section 2 contains a theorem on the finiteness of the Hausdorff dimension of the attractor $\mathcal{A}$ while in Sec. 3 we develop a machinery which allows us to estimate from above in a sufficient exact manner the dimension of the sets that are invariant with respect to $S_t$. In Sec. 4 we give the following esti-
mence of the Hausdorff dimension of an attractor $\mathcal{M}$ of a two-dimensional Navier–Stokes system:

$$\dim_H \mathcal{M} \leq C \exp(C \gamma^4)$$

for zero boundary conditions $u|_{\partial \Omega} = 0$, \(C > 0\) (2)

$$\dim_H \mathcal{M} \leq C / \gamma^n$$

where $\gamma$ is the viscosity coefficient. We note that for any Galerkin approximation of the two-dimensional Navier–Stokes system under periodic boundary conditions, the estimate (3) has been obtained for the first time by Yu. S. Il''yashenko [2].* In Sec. 5 we give the following estimate for the dimension of an attractor for a second-order quasilinear parabolic equation:

$$\dim_H \mathcal{M} \leq C / \gamma^n,$$

where $n$ is the dimension of the space and $\gamma$ is the coefficient of the second derivatives.

1. Existence of a Maximal Attractor

THEOREM 1. Let $S_t : E \to E$ be a semigroup having the following properties: 1) for each $t > 0$ the operator $S_t$ is continuous; 2) for each $t > 0$ the operator $S_t$ maps the bounded sets of $E$ into the precompact sets in $E$; 3) for any $R > 0$ there exist a number $C(R)$ such that if $\|u\| \leq R \quad (l \cdot l = 1 \cdot l_E)$, then $\|S_t u\| \leq C(R)$ for any $t > 0$; 4) there exists a number $C_0$ such that for any $R$ there exists a number $T(R)$ satisfying $\|S_t u\| \leq C_0 \quad \forall t > T(R), \quad \forall u \in E \quad R \leq R$ and $C_0$ does not depend on $R$.

Under the conditions 1)-4), the semigroup $S_t$ has a compact maximal attractor.

Example 1. In the domain $\Omega \times (0, +\infty)$, where $\Omega \in \mathbb{R}^n$, we consider the second-order parabolic equation

$$\frac{\partial u(t, x)}{\partial t} + \gamma \Delta u + b(u, \nabla u) - f(u) - q(x), \quad u|_{\partial \Omega \times (0, +\infty)} = 0,$$

where $\gamma > 0$, $\Delta$ is the Laplace operator, $\nabla u = \text{grad}_x u, q(x) \in C^0(\Omega), \gamma > 0$. One assumes that $f$ and $b$ are smooth functions of their arguments and that

$$|b(u, \xi)| \leq C_1 + C_2(u) |\xi|, \quad \forall \xi \in \mathbb{R}^n, \quad u \in \mathbb{R},$$

$$\|f(u)\| = \sup_{\|v\| = 1} |f(u)| \leq C_3 + C_4 |u|^{q + 1}, \quad w > 0, \quad C_4, C_5 > 0.$$ (6)

THEOREM 2. Assume that inequalities (6), (7) hold. To Eq. (5) there corresponds the semigroup $S_t : E \to E$, where $E = W^{(\theta)}(\Omega) \cap \{u|_{\partial \Omega} = 0\}$, $\ell = 2 - \frac{2}{p}$, $p > n + 2$, which satisfies the conditions 1)-4) of Theorem 1 and has a compact maximal attractor $\mathcal{M} \subset E$.

Example 2. We consider the two-dimensional Navier–Stokes system, written in the form

$$\frac{\partial u(t, x)}{\partial t} + \gamma \nabla u + \sum_{i=1}^{2} u_i \frac{\partial u}{\partial x_i} + \nabla q, \quad q \in (L_2(\Omega))^{2},$$

where $\nabla$ is the orthogonal projection onto the subspace $\mathcal{H}$ obtained by taking the closure in $(L_2(\Omega))^{2}$ of the finite, solenoidal vector fields. In the periodic case, $\mathcal{H}$

*Yu. S. Il' yashenko has informed us that he has recently obtained the estimate (3) for a two-dimensional Navier–Stokes system under periodic boundary conditions.