COMPUTATION OF EXPONENTIAL INTEGRALS

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Let $P \subset \mathbb{R}^d$ be a convex polyhedron and $\mathbf{f}: \mathbb{R}^d \to \mathbb{R}$ a linear function. One studies the computational complexity of the integral $\int_P \exp\{\mathbf{f}(\mathbf{x})\} \, d\mathbf{x}$. It is shown that these integrals satisfy nontrivial algebraic relations, which makes possible the construction of polynomial algorithms for certain polyhedra. Examples are given of the application of exponential integrals to the calculation of volume and nonlinear programming.

Sec. 1. Introduction

Let $\mathbb{R}^d$ be a Euclidean space with the standard scalar product $\langle , \rangle$ and Lebesgue measure $d\mathbf{x}$. In addition, let $P \subset \mathbb{R}^d$ be a $d$-dimensional convex polyhedron. The basic object considered in the present paper is the integral

$$S(P; \mathbf{c}) = \int_P \exp\{\langle \mathbf{c}, \mathbf{x} \rangle\} \, d\mathbf{x} \, , \tag{1.1}$$

and also the integral of somewhat more general form

$$S_p(P; \mathbf{c}) = \int_P \exp\{\langle \mathbf{c}, \mathbf{x} \rangle\} p(\mathbf{x}) \, d\mathbf{x} \, , \tag{1.2}$$

where $\mathbf{c} \in \mathbb{R}^d$, $p: \mathbb{R}^d \to \mathbb{R}$ is a polynomial density.

We shall call integrals of the form (1.1), (1.2) exponential or statistical. We shall be interested in the complexity of computation of the expressions $S(P; \mathbf{c})$, $S_p(P; \mathbf{c})$ and various relations between them permitting one to make such computations.

The interest in exponential integrals is explained by at least the following two circumstances.

Obviously we have

$$S(P; \mathbf{0}) = \text{vol}(P) \, .$$

The problem of computing the volume of a polyhedron is interesting in its own right. It turns out to be useful to imbed it in the more general problem of computing the integrals $S_p(P; \mathbf{c})$, since these expressions are related by a number of algebraic identities which degenerate for $\mathbf{c} = \mathbf{0}$.

Secondly, the computation of exponential integrals can be applied to the solution of problems of polynomial programming. Let $\varphi: \mathbb{R}^d \to \mathbb{R}$ be a polynomial and $\forall \mathbf{x} \in P \quad \varphi(\mathbf{x}) \geq 0$. Then we obviously have

$$\lim_{m \to +\infty} \left( \int_P \varphi^m(\mathbf{x}) \, d\mathbf{x} \right)^{1/m} = \max \{ \varphi(\mathbf{x}) : \mathbf{x} \in P \} \, . \tag{1.3}$$

The computation of integrals of the type of (1.2) lets us get approximate solutions of optimization problems (cf. (5.2)).

We shall assume that the polyhedron $P$ is defined by a system of linear inequalities in $\mathbb{R}^d$, the density $\rho$ and functional $\langle c, \cdot \rangle$ are defined in the standard coordinates of $\mathbb{R}^d$. By the complexity of an algorithm which operates with real numbers one means the number of operations from a given list performed by this algorithm. In our case the list includes the 4 arithmetic actions, the operation of comparison of real numbers, and taking exponentials (iteration of exponentials will actually not be permitted). This model of complexity is not infrequently used in computational geometry (cf. [1]).

Discrete analogs of the integrals (1.1) and (1.2) were considered in [2, 3, 4]; their computation was used in constructing algorithms in some classical problems of combinatorial optimization. The present paper can be considered as the extension of these methods to the continuous case.

The papers [5, 6, 7], etc. on the theory of generalized hypergeometric functions and configurations of hyperplanes also had great influence on the author. The author thanks A. M. Vershik for drawing his attention to this subject and for many fruitful discussions.

We briefly describe the content of the paper.

In Sec. 2 we describe some of the simplest relations between the integrals $S^f(P; c)$, which make their computation possible.

In Sec. 3 we give examples of the computation of integrals (1.1), (1.2).

In Sec. 4 we prove an "additivity theorem" for the integrals $S^f(P; c)$, which actually reduces the computation of (1.2) to the computation of exponential integrals in a neighborhood of the vertices of the polyhedron.

In Sec. 5 we give some consequences of the additivity theorem and also give the general scheme of solution of a problem of the type of (1.3) using exponential integrals.

Sec. 2. Identities for Exponential Integrals

(2.1) Direct Product. Let $P_i \subset \mathbb{R}^d$, $i = 1, \ldots, m$ be a collection of polyhedra, $P_i$; $i = 1, \ldots, m$ be a collection of densities, $P_i \subset \mathbb{R}^d$, $P = P_1 \times \cdots \times P_m \subset \mathbb{R}^d$, where $d = d_1 + d_2 + \cdots + d_m$, $\rho = \rho_1 \times \cdots \times \rho_m$. For $c_i \in \mathbb{R}^d$, let $c = c_1 \oplus c_2 \oplus \cdots \oplus c_m$. Then

$$S^f(P; c) = \prod_{i=1}^m S^f(P_i; c_i).$$

(2.2) Multiplication of a Density by a Polynomial. Let $c_0, c_1, \ldots, c_m \in \mathbb{R}^d$ be a collection of vectors, $\mathbf{t} = (t_1, \ldots, t_m) \in \mathbb{R}^m$ and $c = c_0 + t_1 c_1 + \cdots + t_m c_m$. Then

$$\left. \frac{\partial^{\alpha}}{\partial t_1^{\alpha_1} \cdots \partial t_m^{\alpha_m}} S^f(P; c) \right|_{t=0} = S^f_{\varphi}(P; c_0),$$

where

$$\varphi(x) = \langle c_0, x \rangle \langle c_1, x \rangle^{\alpha_1} \cdots \langle c_m, x \rangle^{\alpha_m}.$$

In general from a polynomial density $\varphi$ one constructs a differential operator $D_{\varphi}$, for which

$$D_{\varphi} S^f(P; c) = S^f_{\varphi}(c).$$

(2.3) Passage to the Section by an Affine Subspace. Let $P$ be a polyhedron in $\mathbb{R}^d$, $f_i : \mathbb{R}^d \to \mathbb{R}$, $i = 1, \ldots, k$ be a collection of affine functions, where $\forall x \in P \exists \varphi(x) > 0$. Let

$$f_i(x) = \frac{\partial}{\partial t_i} f_i(t_1, \ldots, t_k) \left|_{t_1=0, \ldots, t_k=0} \right.$$