STABILITY ESTIMATES OF CONTINUOUS SELECTIONS FOR METRIC ALMOST-PROJECTIONS

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Let \((X, \| \cdot \|)\) be a normed linear space (NLS), \(\varepsilon \geq 0\). A metric almost-projection or \(\varepsilon\)-projection of an element \(x \in X\) on a subset \(M \subset X\) is the set

\[ x_M^\varepsilon = \{ z \in M : \| z - x \| \leq \inf_{y \in M} \| y - x \| + \varepsilon \}. \]

For \(\varepsilon = 0\) we denote this projection by \(x_M\). Let \(Y\) be a subspace of \(X\). We denote by \(C_V(Y)\) the family of all nonempty convex closed subsets of \(Y\) and by \(\mathbb{R}_+ (\mathbb{R}_0)\) the set of positive (nonnegative) real numbers. We put

\[ Z_0(Y) = X \times C_V(Y) \times \mathbb{R}_0, \quad Z(Y) = X \times C_V(Y) \times \mathbb{R}_+. \]

A mapping \(P : Z_0(X) \to C_V(X)\) given by the equality

\[ P(x, M, \varepsilon) = x_M^\varepsilon \]

is called an operator of metric almost-projection.

For \(A, B \in C_V(X)\) we put

\[ AB = \beta(A, B) = \sup_{x \in A} \inf_{y \in B} \| x - y \|, \]

\[ \alpha(A, B) = \max(A, B) \cdot \max(B, A). \]

We use the abridged notation \(xy = \| x - y \|\). The distance \(\alpha\) is called a Hausdorff metric on \(C_V(X)\). Note that, although the deviation \(\beta\) is not symmetric, it satisfies the triangular inequality

\[ \beta(A, B) \leq \beta(A, C) + \beta(C, B). \]

The metric \(\rho\) on the set \(Z_0(Y)\) can be introduced by

\[ \rho(v, w) = |\varepsilon - \delta| + 2xy + 3\alpha(M, N), \]

where \(v = (x, M, \varepsilon)\), \(w = (y, N, \delta)\).

Recently, much attention was paid to the question of the existence of continuous and Lipschitz continuous selections for the operator of the metric projection \(x \to x_M\) (see, for example, [1-3] and references given therein). A selection of a set-valued mapping \(F(x)\) is a single-valued mapping \(s(x)\) such that \(s(x) \in F(x)\) for any \(x\). The continuous selection of the mapping \(x \to x_M^\varepsilon\), \(\varepsilon > 0\), is investigated in [4, 5].

In the present paper we prove the existence and obtain stability estimates for the continuous selection of the operator \(P\) which acts on the space \((Z(Y), \rho)\). We use the stability estimate for the operator \((x, M) \to x_M\) with the assumption that the space \(X\) is uniformly convex (Theorem 2). What is more, the key role in the proofs is played by the inequality obtained in [6], which, in particular, implies that the operator \(P\) is pointwise Lipschitz in the Hausdorff metric.

The character of the continuity of the selection under consideration depends on the choice of the subspace \(Y \subset X\). Thus, for a Banach space \(Y\) admitting the equivalent renormalization into a uniformly convex space \((Y, \| \cdot \|')\) there exists a selection such that the distances between its values are estimated by a function inverse to the convexity module of the space \((Y, \| \cdot \|')\) (Theorem 3). But if \(Y\) is a finite-dimensional subspace, then one can prove the existence of a homogeneous pointwise Lipschitz selection with the help of Steiner points (Theorem 4). In the case when no supplementary restrictions are imposed on the Banach subspace \(Y\), we can prove only the existence of a continuous selection (Theorem 1).

Let us recall that the convexity module of the space \(X\) is the function

\[ \varphi(t) = \inf \left\{ 1 - \frac{\|x + y\|}{2} : \|x\| = \|y\| = 1, \ xy \geq t \right\}. \]
defined for \( t \in [0, 2] \). The space \( X \) is called uniformly convex, if \( \varphi(t) > 0 \) for any \( t > 0 \). But we need the function inverse to \( \varphi(t) \):

\[
\varphi^{-1}(s) = \sup\left\{ xy : 1 - \frac{\|x + y\|}{2} \leq s, \|x\| = \|y\| = 1 \right\}, \quad s \in [0, 1].
\]

For \( s \geq 1 \) this function is defined by the continuity: \( \varphi^{-1}(s) = 2 \).

We denote by \( \text{diam} \ A \) the diameter of the set \( A \). Let \( B(x, r) = \{ y \in X : xy \leq r \} \), and let \( S(x, r) \) be the boundary of \( B(x, r) \). If it is necessary to express explicitly the norm, by which the almost-projection \( x_M^e \) is constructed, we write \( P(x, M, \|\cdot\|, \varepsilon) \). In the sequel by \( Z(Y) \) we mean the metric space \((Z(Y), \rho)\).

In [6] for points \( v = (x, M, \varepsilon), w = (y, N, \delta) \in Z(X) \) we obtained the estimate

\[
\alpha(P(v), P(w)) \leq \left( \frac{2E}{\mu} + 1 \right) \rho(v, w),
\]

where \( E = \min(xM, yN) \), \( \mu = \max(\varepsilon, \delta) \). Since \( E/\mu \leq xM/\varepsilon \), it follows from (1) that the operator \( P \) satisfies the Lipschitz condition in the Hausdorff metric at each point \( v \in Z(X) \). But it follows immediately from the definitions that if a set-valued mapping is continuous by Hausdorff, then it is lower semicontinuous. Consequently, the operator \( P \) is semicontinuous on \( Z(X) \). But since, in the case considered, \( P(v) \in C_\nu(X) \), we see that the known Michael theorem [7] can be applied to \( P \), that is, the following theorem is valid.

**Theorem 1.** Let \( Y \subset X \) be a Banach subspace. Then the operator \( P \) has a continuous selection on the set \( Z(Y) \).

We pass to stability estimates for a continuous selection. For this purpose we need a number of auxiliary statements.

Let \( \varphi(t) \) be the convexity module of the space \( X \). For \( a, t \geq 0, a + t > 0 \), we put

\[
\Phi(a, t) = (a + t)^{-1}\left( \frac{t}{a + t} \right),
\]

\[
\Psi(a, t) = (a + t)^{-1}\left( \frac{2t}{a + t} \right).
\]

It is clear that

\[
\Psi(a, t) = \begin{cases} \Phi(a-t, 2t), & a \geq t, \\ 2(a+t), & a \leq t. \end{cases}
\]

**Lemma.** Let \( X \) be an arbitrary (not necessarily uniformly convex) NLS. If \( a + t \leq b + s \) and \( t \leq s \), then the inequalities

\[
\Phi(a, t) \leq \Phi(b, s), \quad \Psi(a, t) \leq \Psi(b, s)
\]

are true. In particular, the functions \( \Phi(a, t) \) and \( \Psi(a, t) \) increase with respect to each of the variables \( a, t \).

**Proof.** Let us prove the first inequality. For this purpose, obviously, it is sufficient to establish the inequality \((a + t)xy \leq \Phi(b, s)\) under the condition that points \( x, y \in S(0, 1) \) are chosen so that the relation \( 1 - \|x + y\|/2 \leq t/(a + t) \) holds. In other words, one has to prove that the conditions \( x, y \in S(0, a + t) \) and \( a + t - \|x + y\|/2 \leq t \) imply the inequality \( xy \leq \Phi(b, s) \).

Denote \( z = (x + y)/2 \), and let a point \( q \) lie on the intersection of a ray \( L \) passing through a point \( z \) with its beginning at the origin. Since \( a + t \leq b + s \), we have \( \|z\| \leq \|q\| \). Let us displace the ball \( B(0, a + t) \) parallel to itself along the ray \( L \) so that it is tangent on the inside to the sphere \( S \). It is clear that the point \( q \) is a point of tangency. With this displacement the points \( x, y, z \) are removed to points \( x_1, y_1, z_1 \), respectively. So, we have \( qz_1 = t \leq s \). If the points \( x_1, y_1 \) do not lie on the sphere \( S \), then we move the interval \([x_1, y_1]\) so that its center \( z_1 \) is on the ray \( L \) and the ends are on the sphere \( S \). We denote by \( x_2, y_2, z_2 \) new positions of these points. Since \( z_2q \leq t \leq s \), we have \( x_2y_2 \leq \Phi(b, s) \), hence, in view of the equality \( xy = x_2y_2 \) the first inequality of (2) holds.