INEQUALITIES FOR EIGENVALUES OF THE POLYHARMONIC OPERATOR $\Delta^p$

CHEN ZUCHI (陈祖璞)
(University of Science & Technology of China)

Abstract

In this paper we consider the bounds of the eigenvalues for a class of polyharmonic operators and obtain the bounds for $(n+1)$-th eigenvalue in terms of the first $n$ eigenvalues. Those estimates do not depend on the domain in which the problem is considered.

Introduction

Let $u$ be a solution of the polyharmonic equation

$$\Delta^p u - \mu u = 0, \quad p = 2^l, \quad l = 1, 2, \ldots$$

in a bounded domain $\Omega \subset \mathbb{R}^m, \; m > 1$ with sufficiently smooth boundary $\partial \Omega$, subject to the homogeneous boundary conditions

$$u - \frac{\partial u}{\partial n} - \cdots - \frac{\partial^{p-1} u}{\partial n^{p-1}} = 0 \text{ on } \partial \Omega,$$

$n$ being an outward unit normal. We assume that the spectrum, i.e. those values of $\mu$ for which a non-trivial solution exists, is discrete. Ordering the eigenvalues in ascending magnitude, $0 < \mu_1 < \mu_2 < \cdots < \mu_n < \cdots$, and denoting the corresponding eigenfunctions by $u_1, u_2, \ldots, u_n, \ldots$, we shall obtain inequalities among the first $n+1$ eigenvalues, which are independent of the domain.

L. E. Payne, G. Polya and H. F. Weinberger showed that, for the harmonic operator $\Delta$ and a domain in the plane$^{[1]}$,

$$\mu_{n+1} \leq \mu_n + \frac{2}{n} \sum_{i=1}^{n-1} \mu_i \leq \frac{3}{2} \mu_n, \quad n = 1, 2, \ldots \tag{3}$$

Inequality (3) was improved and extended to $\mathbb{R}^m, \; m > 2$ by M. H. Protter and G. N. Hile$^{[2]}$. They showed that the eigenvalues for the harmonic operator $\Delta$ satisfy the inequality

$$\sum_{i=1}^{n} \frac{\mu_i}{\mu_{n+1} - \mu_i} \geq \frac{mn}{4}, \tag{4}$$

which gives an implicit bound for $\mu_{n+1}$ in terms of the preceding eigenvalues. Replacing each $\mu_i$ with $\mu_n$ in the denominators on the left side of (4), and then working a solution for $\mu_{n+1}$ we get the explicit bound for $\mu_{n+1}$ as follows:

$$\mu_{n+1} \leq \mu_n + \frac{4}{mn} \sum_{i=1}^{n} \mu_i. \tag{5}$$

For the case $p=2$ ($l=1$), i.e. that of biharmonic operator $\Delta^2$, G. N. Hile and

$^*$ Received October 22, 1985.
R. Z. Yeh obtained the upper estimates for $\mu_{n+1}$ in a bounded domain in $\mathbb{R}^m (m \geq 2)$:

$$\mu_{n+1} \leq \mu_n + \frac{8(m+2)}{m^2 a_m} \sum_{i=1}^{n} \mu_i \leq \left( \frac{m+4}{m} \right)^2 \mu_n,$$

and the improved form for both implicit and explicit bounds:

$$\sum_{i=1}^{n} \frac{\sqrt{\mu_i}}{\mu_{n+1} - \mu_i} \leq \frac{m^2 a_m^2}{8(m+2)} \left( \sum_{i=1}^{n} \mu_i \right)^{-\frac{1}{2}},$$

$$\mu_{n+1} \leq \mu_n + \frac{8(m+2)}{m^2 a_m^2} \left( \sum_{i=1}^{n} \mu_i \right)^{\frac{3}{2}} \left( \sum_{i=1}^{n} \sqrt{\mu_i} \right).$$

In this paper we consider the polyharmonic operators $\mathcal{A}(p=2^l, l=1, 2, \ldots)$ in a bounded domain in $\mathbb{R}^m (m \geq 1)$ with sufficiently smooth boundary, and get the upper bounds for $\mu_{n+1}$ in term of the first $n$ eigenvalues. The inequalities we get (See Theorem 1 and Theorem 2) include the case of harmonic and biharmonic operators, i.e. inequalities (3) to (8) are included.

§ 1. Inequalities for $\mu_{n+1}$

Let $\Omega$ be a bounded domain in $\mathbb{R}^m, m \geq 1$, with sufficiently smooth boundary $\partial \Omega$. Ordering the eigenvalues of the problem

$$\mathcal{A} u - \mu u = 0 \quad \text{in} \quad \Omega, \quad p=2^l, l=1, 2, \ldots$$

$$u = \frac{\partial u}{\partial n} = \cdots = \frac{\partial^{p-1} u}{\partial n^{p-1}} = 0 \quad \text{on} \quad \partial \Omega, \quad n \text{ being an outward unit normal},$$

by $0 < \mu_1 \leq \mu_2 \leq \cdots \leq \mu_n \leq \cdots$ with corresponding eigenfunctions $u_1, u_2, \cdots, u_n, \cdots$ normalized so that

$$\int_{\Omega} u_i u_j = \delta_{ij}, \quad i, j = 1, 2, \ldots,$$

replacing $u$ by $u_i$ in (1.1), then multiplying by $u_i$ and integrating by parts repeatedly, we get

$$\mu_i = \int_{\Omega} \left( \sum_{j=1}^{n} \mu_j \right)^2 dx.$$

In the sequel, we will use this equality many times without mentioning it again. We also use the symbols $\sum_{i=1}^{n}$ for $\sum_{i=1}^{n}$ and $\int_{\partial} \cdots dx$.

For establishing our results, we give two lemmas in advance. Following Protter and Hile we introduce $n$ trial functions

$$\varphi_i = a_i u_i - \sum_{j=1}^{n} a_{ij} u_j, \quad i = 1, 2, \cdots, n, \quad x = (x_1, \cdots, x_n)$$

with constants given by

$$a_{ij} = \int_{\Omega} x_i u_j u_i, \quad i, j = 1, 2, \cdots, n.$$

Thus, each $\varphi_i$ is orthogonal to $u_1, u_2, \cdots, u_n$; and since

$$\frac{\partial \varphi_i}{\partial n} = \cdots = \frac{\partial^{p-1} \varphi_i}{\partial n^{p-1}} = 0,$$