INEQUALITIES FOR EIGENVALUES OF THE
POLYHARMONIC OPERATOR \( \Delta^p \)

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Abstract

In this paper we consider the bounds of the eigenvalues for a class of polyharmonic operators and obtain the bounds for \((n+1)\)th eigenvalue in terms of the first \(n\) eigenvalues. Those estimates do not depend on the domain in which the problem is considered.

Introduction

Let \( u \) be a solution of the polyharmonic equation
\[
\Delta^p u - \mu u = 0, \quad p = 2^l, \quad l = 1, 2, \ldots
\]
in a bounded domain \( \Omega \subset \mathbb{R}^m, \quad m > 1 \) with sufficiently smooth boundary \( \partial \Omega \), subject to the homogeneous boundary conditions
\[
u = \frac{\partial u}{\partial n} = \cdots = \frac{\partial^{p-1} u}{\partial n^{p-1}} = 0 \quad \text{on} \quad \partial \Omega,
\]
where \( n \) is an outward unit normal. We assume that the spectrum, i.e., those values of \( \mu \) for which a non-trivial solution exists, is discrete. Ordering the eigenvalues in ascending magnitude, \( 0 < \mu_1 < \mu_2 < \cdots < \mu_n < \cdots \), and denoting the corresponding eigenfunctions by \( u_1, u_2, \ldots, u_n, \ldots \), we shall obtain inequalities among the first \( n+1 \) eigenvalues, which are independent of the domain.

L. E. Payne, G. Polya and H. F. Weinberger showed that, for the harmonic operator \( \Delta \) and a domain in the plane\(^{[1]} \),
\[
\mu_2 \leq \mu_n + 2 \sum_{i=1}^{n-1} \mu_i < 3 \mu_n, \quad n = 1, 2, \ldots.
\]
Inequality (3) was improved and extended to \( \mathbb{R}^m, \quad m > 2 \) by M. H. Protter and G. N. Hile\(^{[2]} \). They showed that the eigenvalues for the harmonic operator \( \Delta \) satisfy the inequality
\[
\sum_{i=1}^{n} \frac{\mu_i}{\mu_{n+1} - \mu_i} \geq \frac{nn}{4},
\]
which gives an implicit bound for \( \mu_{n+1} \) in terms of the preceding eigenvalues. Replacing each \( \mu_i \) with \( \mu_n \) in the denominators on the left side of (4), and then working a solution for \( \mu_{n+1} \) we get the explicit bound for \( \mu_{n+1} \) as follows:
\[
\mu_{n+1} \leq \mu_n + \frac{4}{mn} \sum_{i=1}^{n} \mu_i.
\]

For the case \( p = 2 \) (\( l = 1 \)), i.e., that of biharmonic operator \( \Delta^2 \), G. N. Hile and

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R. Z. Yeh obtained the upper estimates for $\mu_{n+1}$ in a bounded domain in $\mathbb{R}^n (m \geq 2)$:

$$
\mu_{n+1} \leq \mu_n + \frac{8(m+2)}{m^2 n^2} \sum_{i=1}^{n} \mu_i \quad \text{for } m \geq 2,
$$

and the improved form for both implicit and explicit bounds:

$$
\sum_{i=1}^{n} \frac{\sqrt{\mu_i}}{\mu_{n+1} - \mu_i} \geq \frac{m^2 n^2}{8(n+2)} \left( \sum_{i=1}^{n} \mu_i \right)^{-\frac{1}{2}},
$$

$$
\mu_{n+1} \leq \mu_n + \frac{8(m+2)}{m^2 n^2} \left( \sum_{i=1}^{n} \mu_i \right)^{\frac{1}{2}} \left( \sum_{i=1}^{n} \sqrt{\mu_i} \right).
$$

In this paper we consider the polyharmonic operators $\mathcal{A}(p=2^l, l=1, 2, \ldots)$ in a bounded domain in $\mathbb{R}^n (m>1)$ with sufficiently smooth boundary, and get the upper bounds for $\mu_{n+1}$ in terms of the first $n$ eigenvalues. The inequalities we get (See Theorem 1 and Theorem 2) include the case of harmonic and biharmonic operators, i.e. inequalities (3) to (8) are included.

§ 1. Inequalities for $\mu_{n+1}$

Let $\Omega$ be a bounded domain in $\mathbb{R}^n, m>1$, with sufficiently smooth boundary $\partial \Omega$. Ordering the eigenvalues of the problem

$$
\mathcal{A} u - \mu u = 0 \quad \text{in } \Omega, \quad p=2^l, \ l=1, 2, \ldots
$$

$$
u = \frac{\partial u}{\partial n} = \cdots = \frac{\partial^{p-1} u}{\partial n^{p-1}} = 0 \quad \text{on } \partial \Omega, \quad n \text{ being an outward unit normal,}
$$

by $0 < \mu_1 < \mu_2 < \cdots < \mu_n$ with corresponding eigenfunctions $u_1, u_2, \ldots, u_n, \ldots$ normalized so that

$$
\int_\partial u_i u_j = \delta_{ij}, \quad i=1, 2, \ldots,
$$

replacing $u$ by $u_i$ in (1.1), then multiplying by $u_i$ and integrating by parts repeatedly, we get

$$
\mu_i = \int_\partial (d^2 u_i)^2 \ dx.
$$

In the sequel, we will use this equality many times without mentioning it again. We also use the symbols $\Sigma_i$ for $\sum_{i=1}^{n}$ and $\int_\partial \cdots \ dx$.

For establishing our results, we give two lemmas in advance. Following Protter and Hile we introduce $n$ trial functions

$$
\varphi_i = x_{i} u_i - \sum_{j=1}^{n} a_{ij} u_j, \quad i = 1, 2, \ldots, n, \quad x = (x_1, \ldots, x_n)
$$

with constants given by

$$
a_{ij} = \int_\partial x_i u_j u_i, \quad i=1, 2, \ldots, n.
$$

Thus, each $\varphi_i$ is orthogonal to $u_2, u_3, \ldots, u_n$; and since

$$
\varphi_i = \frac{\partial \varphi_i}{\partial n} = \cdots = \frac{\partial^{p-1} \varphi_i}{\partial n^{p-1}} = 0
$$

and