HOLDER ESTIMATE FOR THE SOLUTION OF DEGENERATE PARABOLIC EQUATIONS*

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Abstract

Consider the degenerate parabolic equations of the type

\[ u_t = \text{div} \left( a(x, t, u, Du) + b(x, t, u, Du) \right) \]

which is of the same nature as

\[ u_t = \text{div} \left| Du \right|^{2p} + |Du|^{p+2} \quad (p > 2). \]

This paper is to study the \( C^{1+\alpha}_{1-\alpha} \)-Hölder continuity of a class of degenerate parabolic equations and the existence and uniqueness of the initial boundary value problem.

§ 1. Introduction and Main Results

The main purpose of this paper is to establish the inner \( C^{1+\alpha}_{1-\alpha} \) estimations of the degenerate parabolic equation

\[ u_t = \text{div} (A(x, t, u, Du)) + b(x, t, u, Du) \quad (1.1) \]

and to prove the existence and uniqueness of the initial value problem

\[ \begin{cases} u_t = \text{div} (A(x, t, u, Du)) + b(x, t, u, Du) & \text{in } Q_T, \\ u = g(x, t) & \text{on } \partial Q_T, \end{cases} \quad (1.2) \]

where \( Q_T = \Omega \times (0, T) \) and \( \Omega \) is a bounded open smooth subset of \( \mathbb{R}^N \), \( A(x, t, u, P) = (a^1(x, t, u, P), \ldots, a^N(x, t, u, P)) \) is a map from \( Q_T \times \mathbb{R}^N \) into \( \mathbb{R}^N \), and \( b(x, t, u, P) \) maps \( Q_T \times \mathbb{R} \times \mathbb{R}^N \) into \( \mathbb{R} \).

We assume that \( A \) and \( b \) satisfy the following assumptions:

\((A1)\) for every \( \xi \in \mathbb{R}^N \) and \( (x, t, u, P) \in Q_T \times \mathbb{R} \times \mathbb{R}^N \),

\[ \lambda V(|P|)|\xi|^2 \leq \sum_{i=1}^{N} \frac{\partial a^i(x, t, u, P)}{\partial P_i}(x, t, u, P)\xi_i\xi_i \leq \Lambda V(|P|)|\xi|^2, \quad (1.3) \]

where \( \lambda \) and \( \Lambda \) are positive constants and \( V(\cdot) \) is a nondecreasing function which has the following properties:

(a) there exists a constant \( p > 0 \) such that

\[ s^p \leq V(s) \leq \lambda (1+s^p), \quad \forall s \in (0, +\infty); \quad (1.4) \]

(b) there exists a constant \( \gamma \in (0, 1) \) such that

\[ V(s) \leq 2^p V(\gamma s). \quad (1.5) \]

\((A2)\) for every \( (x, t, u, P) \in Q_T \times \mathbb{R} \times \mathbb{R}^N \), \( A(x, t, u, P) \) satisfies

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and $b(x, t, u, P)$ satisfies
\[ |b(x, t, u, P)| \leq A(1 + |P|^\gamma V(|P|)). \] (1.7)

The corresponding elliptic problems have been considered by Uhlenbeck [18], Ural'tzeva [14], Evans [6], DiBenedetto [5], Tolsdorf [12], etc. The continuity and Hölder continuity of the spatial gradient of the solution of (1.1) in the case $b = 0$ and $A = |Du|^\gamma$ have been proved by Alikakos & Evans [10] and Chen [4] respectively. In their case, they can get the boundedness of the gradient directly by Moser iteration, but in our case, it is hard to obtain the boundedness of the spatial gradient of the solution without the estimate of continuity. In order to obtain the boundedness of the spatial gradient, we shall estimate the continuity of the solution.

**Notations.**

\[ D_t^u = (u_{x_1}, u_{x_2}, \ldots, u_{x_d}), \quad D^u = (u_{x_1}, u_{x_2}, \ldots, u_{x_d}), \]
\[ B_0 = \{ x \in \mathbb{R}^n \mid |x| < R \}, \quad C_0(\mu, R) = z + B_R \times (-\mu R^{n+2}, 0); \]

for every subset $D$ of $Q_T$,

\[ \partial_p D = \text{parabolic boundary of } D, \]
\[ \partial D = \text{parabolic boundary of } D, \]
\[ u|_{\partial D} = \left\{ \begin{array}{ll}
\int_0^t \int_D |D^2 u(x, s)| ds dx + \int_D |u(x, s)| ds dx \leq C & (1.3) \\
\end{array} \right. \]
\[ \|u\|_{L^p(D)} = \left\{ \begin{array}{ll}
\int_D |u(x)| dx & (1.4) \\
\end{array} \right. \]
\[ u_{i+1}(D) = \{ u \in L^p(D) \mid Du \in L^p(D) \}, \]
\[ U(D) = \left\{ u \in L^p(D) \mid \text{ and } b_i(x, t, u, Du) \text{ satisfying ((A1)), ((A2)), and (1.4)}, \text{ such that } u \text{ satisfies (1.1)} \right\} \]

where
\[ s + s^p \leq V(s), \quad \forall s \in (0, +\infty), \quad (1.4), \]
\[ u_t = \text{div } A(x, t, u, Du) + b_i(x, t, u, Du). \quad (1.1), \]

**Definition 1.1.** A weak solution of (1.1) means a function $u \in L^{p+\gamma}_0(Q_T)$ with $Du \in L^{p+\gamma}_0(Q_T)$ satisfying the integral identity
\[ \left[ -uf_t + A(x, t, u, Du) \cdot Df - b(x, t, u, Du)f \right] dx dt = 0 \quad (1.8) \]

for every $f \in C^0_c(Q_T)$.\n
**Definition 1.2.** A weak solution of (1.1) and (1.2) means a function $u \in C^0(Q_T)$ which is not only a weak solution of (1.1), but also satisfies (1.2).

Our main results are the following:

**Theorem 1.1.** Let $u$ be a weak solution of (1.1) and assume $u$ can be locally constructed as the weak $L^{p+\gamma}_0(Q_T)$ limit of a uniformly bounded set $\{u_i\}$ in $U(Q_T)$. Then for every compact subdomain $Q' \subset Q_T$ there exist constants $C > 0$ and $\alpha \in (0, 1)$ depending only upon $\lambda, A, N, p, \sup_{n=0}^\infty |u_n|_{L^p(Q_T)}, \gamma$, and $\text{dist}(Q' \subset Q_T)$, such that