Joint error distributions are estimated for sampled-data measurements.

Various quantities in engineering are often measured by sampled-data methods. These methods are conveniently described by the following scheme. The sampling points are located on a straight line at a distance \( h \) from one another. We measure the length of the interval \([0, \varphi]\) on the same line. The estimate of the length of the interval \([0, \varphi]\) is computed as \( \hat{\varphi} = N\varphi h \), where \( N\varphi \) is the number of sampling points captured by the interval \([0, \varphi]\). The measurement error is \( \Delta\varphi = \varphi - \hat{\varphi} = \varphi - N\varphi h \).

Suppose that we simultaneously measure \( k \) quantities \( \varphi_1, \varphi_2, \ldots, \varphi_k \), the lengths of the intervals \([0, \varphi_1]\), \([0, \varphi_2]\), \ldots, \([0, \varphi_k]\). Denote by \( \Delta\varphi_i \) the errors in these measurements; \( A_i, i = 1, \ldots, k \), are some constants. We are often faced with the problem of estimating the error distribution when the sum \( \sum_{i=1}^{k} A_i\varphi_i \) is estimated by \( \sum_{i=1}^{k} A_i\hat{\varphi}_i \), i.e., the problem of estimating the probability

\[
P \left( \left| \sum_{i=1}^{k} A_i\Delta\varphi_i \right| > x \right)
\]

with \( x > 0 \), and also the problem of estimating the joint distribution of the errors \( \Delta\varphi_i, i = 1, \ldots, k \), i.e.,

\[
P : \Delta\varphi_i > x, i = 1, \ldots, k
\]

We will solve both these problems. We will also consider the case of independent measurements with the averaged estimators

\[
\varphi_i = \frac{1}{n} \sum_{j=1}^{n} \varphi_{ij}
\]

Note that \( n \) is not large enough to permit using the limit estimates that follow from the central limit theorem.

We will derive exact estimates for any \( n \). For \( \varphi \) we define \( \delta_{\varphi} \) by the equality \( \varphi = Nh + \delta_{\varphi} \), where \( N \) is the greatest integer such that \( Nh < \varphi, 0 \leq \delta_{\varphi} \leq h \).

**Assumption 1.** \( \delta_{\varphi_1}, \ldots, \delta_{\varphi_k} \) are independent random variables, each uniformly distributed on \([0, h]\). This means that the observations are sampled independently of one another.

Denote by \( \xi_1 \) the distance from the point 0 to the first sampling point located to the right of 0; by \( \xi_{\varphi_i} \) the distance from the point \( \varphi_i \) to the last sampling point on the interval \([0, \varphi_i]\), \( i = 1, \ldots, k \).

**Assumption 2.** \( \xi_1 \) is a random variable uniformly distributed on \([0, h]\) and independent of \( \delta_{\varphi_1}, \ldots, \delta_{\varphi_k} \). It is easy to see that the error of measuring the angle \( \varphi_i \) equals \( \Delta\varphi_i = \xi_1 + \xi_{\varphi_i} - h \) and

\[
\xi_{\varphi_i} = \begin{cases} 
  h - \xi_1 + \delta_{\varphi_i}, & \text{if } \delta_{\varphi_i} < \xi_1; \\
  \delta_{\varphi_i} - \xi_1, & \text{if } \delta_{\varphi_i} \geq \xi_1.
\end{cases}
\]
Let us find the conditional distribution of \( \xi_{\varphi_i} \) given \( \xi_1 \). Note that for a fixed \( \xi_1 \)

\[
P \{ \delta_{\varphi_i} < \xi_1 \} = \xi_1/h, \quad P \{ \delta_{\varphi_i} > \xi_1 \} = 1 - \xi_1/h.
\]

Thus, for \( 0 \leq x \leq h \),

\[
P_1 : \xi_{\varphi_i} < x \} = P \{ \xi_{\varphi_i} < x, \delta_{\varphi_i} < \xi_1 \} + P \{ \xi_{\varphi_i} < x, \delta_{\varphi_i} > \xi_1 \} =
\]

\[
= P \{ \delta_{\varphi_i} < x + \xi_1 - h, \delta_{\varphi_i} < \xi_1 \} + P \{ \delta_{\varphi_i} < \xi_1 + x, \delta_{\varphi_i} > \xi_1 \}.
\]

Since

\[
P \{ \delta_{\varphi_i} < \xi_1 + x, \delta_{\varphi_i} > \xi_1 \} = \frac{x}{h}, \quad x \leq \xi_1 - \xi_1;
\]

\[
P \{ \delta_{\varphi_i} < x + \xi_1 - h, \delta_{\varphi_i} < \xi_1 \} = \begin{cases} 0, & x \leq \xi_1 - \xi_1; \\ \frac{x + \xi_1 - h}{h}, & x > \xi_1 - \xi_1; \end{cases}
\]

we have \( P\{\xi_{\varphi_i} < x\} = x/h \) for \( 0 \leq x \leq h \) and this probability is independent of the value of \( \xi_1 \), i.e., the random variables \( \xi_1 \) and \( \xi_{\varphi_i} \) are independent.

**Lemma 1.** The random vector

\[
\Delta \varphi = \{ \Delta \varphi_i, i = 1, k \}
\]

consists of elements representable in the form

\[
\Delta \varphi_i = \eta_1 + \eta_{\varphi_i}, \quad i = 1, k,
\]

where \( \eta_1 \) and \( \eta_{\varphi_i} \) are random variables uniformly distributed on \([-h/2, h/2]\), \( \eta_{\varphi_i} \) are independent of one another and of \( \eta_1 \) \((M_{\eta_1} = M_{\eta_{\varphi_i}} = 0, D_{\eta_1} = D_{\eta_{\varphi_i}} = h^2/3\)).

The lemma follows from the fact that

\[
\Delta \varphi_i = \xi_1 + \xi_{\varphi_i} - h = \xi_1 - \frac{h}{2} + \xi_{\varphi_i} - \frac{h}{2}.
\]

Denote \( \eta_1 = \xi_1 - h/2 \), and \( \eta_{\varphi_i} = \xi_{\varphi_i} - h/2 \). Then from the above argument it follows that \( \eta_1 \) and \( \eta_{\varphi_i} \) satisfy all the assertions of the lemma.

**Definition 1.** The random vector \( \vec{\xi} = (\xi_1, ..., \xi_k) \) is called strictly sub-Gaussian if \( M\xi = 0 \) and for every \( \vec{\lambda} \in \mathbb{R}^k \) we have the inequality

\[
M \exp \{ (\vec{\lambda}, \vec{\xi}) \} \leq \exp \left( \frac{1}{2}(B\vec{\lambda}, \vec{\lambda}) \right),
\]

where \( B \) is the covariance matrix of the vector \( \vec{\xi} \) \([2, 4]\).

**Remark.** For \( k = 1 \) the random variable \( \xi \) is called strictly sub-Gaussian \([1, 3]\). It satisfies the inequality

\[
M \exp \{ \lambda \xi \} \leq \exp \left( \frac{\lambda^2}{2} M \xi^2 \right)
\]

for all \( \lambda \in \mathbb{R}^1 \).

**Lemma 2.** The random vector \( \Delta \varphi' = \{ \Delta \varphi_i, i = 1, ..., k \} \) is strictly sub-Gaussian.

**Proof.**

\[
M \exp \{ (\vec{\lambda}, \Delta \varphi) \} = M \exp \left( \sum_{i=1}^{k} \lambda_i (\eta_1 + \eta_{\varphi_i}) \right) =
\]