MULTITERMINAL XCUT PROBLEMS

Refael HASSIN

School of Mathematics, Tel-Aviv University, Tel-Aviv 69978, Israel

Abstract

An i–j xcut of a set $V = \{1, \ldots, n\}$ is defined to be a partition of $V$ into two disjoint nonempty subsets such that both $i$ and $j$ are contained in the same subset. When partitions are associated with costs, we define the i–j xcut problem to be the problem of computing an i–j xcut of minimum cost. This paper contains a proof that the $\binom{n}{2}$ minimum xcut problems have at most $n$ distinct optimal solution values. These solutions can be compactly represented by a set of $n$ partitions in such a way that the optimal solution to any of the problems can be found in $O(n)$ time. For a special additive cost function that naturally arises in connection to graphs, some interesting properties of the set of optimal solutions that lead to a very simple algorithm are presented.

1. Introduction

Let $V = \{1, \ldots, n\}$ be a given set. A cut of $V$ is a partition $(S, T)$ of $V$ into two disjoint nonempty subsets $S$ and $T$. For each cut $(S, T)$ let $c(S, T)$ be its cost (or value), where $c$ is an arbitrary real function defined on the cuts of $V$. In this paper, we assume that $c$ is symmetric, i.e. $c(S, T) = c(T, S)$. For a given pair of distinct elements $i, j \in V$, an i–j cut is a cut $(S, T)$ such that either $i \in S$ and $j \in T$, or $i \in T$ and $j \in S$. In this paper, we define an i–j xcut to be a cut $(S, T)$ such that either $i, j \in S$ or $i, j \in T$. The minimum s–t cut (xcut) problem is to find the s–t cut (xcut) of minimum cost. An example of such a problem is a scheduling problem on two machines with a constraint that a given pair of jobs must be processed on the same machine. A possible objective function is to minimize the makespan. In this case, $c(S, T) = \max\{c(S), c(T)\}$, where $c(X) = \sum_{i \in X} t_i$ and $t_i$ is the processing requirement for job $i$. A multiterminal problem requires solving a set of i–j problems for different i–j pairs. As an example of a multiterminal xcut problem, consider a communication network where an option exists to connect one pair of users by a line of a very high capacity, and it is desired to assess the effects of applying this option to any member of a given set of pairs. For any given realization of the option, the edge connectivity of the resulting network will be equal to the minimum xcut with respect to the pair chosen to be connected by the high-capacity line. To choose the best option, one has to solve a multiterminal xcut problem.

Gomory and Hu [3] analyzed the multiterminal cut problem for a special additive cost function. Since problems with such costs usually arise in connection with graphs, we call them graphic problems. Gomory and Hu proved that the $\binom{n}{2}$
graphic cut problems have at most \( n - 1 \) distinct solutions. They also constructed a data structure (a cut tree) that stores these values in such a way that allows the determination of the solution to any of the problems in linear time. They proved that a cut tree can be constructed from a set of \( n - 1 \) noncrossing cuts, and use this property to compute a cut tree by solving only \( n - 1 \) cut problems. These results were extended by Granot and Hassin [4] to networks with node-capacities and by Gusfield and Naor [5] to a compact representation of all of the minimum cuts for each pair of elements. Hassin [6, 7], and Cheng and Hu [1] extended Gomory and Hu's results by showing that the multiterminal problem has a most \( n - 1 \) distinct solutions for arbitrary cost functions. Hassin's result comes from a general theorem that also supplies a compact representation to the solutions (called a maximum solution basis), generalizing the concept of a cut tree (see section 2).

In this paper, we obtain related results for the multiterminal xcut problem. In section 3, we characterize the structure of the solution bases and show that the number of distinct solutions is at most \( n \). In section 4, we consider multiterminal problems consisting of a mixed set of cut and xcut problems and characterize their solution bases. Finally, in section 5, we consider graphic multiterminal problems and describe a particularly simple algorithm for this special case.

2. Solution bases and the 2-forest matroid

Let \( c \) be a real valued function on a set \( X \). Let \( X_i, i \in I \), be nonempty subsets of \( X \), where \( I \) is an index set with finite cardinality. Define the cost (or value) of \( X_i \) by

\[
 c_i = \min \{ c(x) | x \in X_i \}. 
\]

Let \( M(I) \) be the number of distinct values of \( c_i, i \in I \). Let \( a_i(x) = 1 \) if \( x \in X_i \) (i.e. \( x \) is \( i \)-feasible) and \( a_i(x) = 0 \) otherwise.

For our purpose, the set \( X \) is the set of candidate solutions to a family of problems, and \( X_i \) is the set of \( i \)-feasible solutions, \( i \in I \). The binary matrix \( A = (a_i(x)) \), called the solution matrix, defines the feasibility relations, and the cost \( c_i \) is the minimum cost of an \( i \)-feasible solution (i.e. the cost of an \( i \)-optimal solution).

Call a set \( I' \subseteq I \) dependent if there exists \( I'' \subseteq I' \) such that \( \sum_{i \in I'} a_i(x) = 0 \mod 2 \) \( \forall x \in X \). Otherwise, \( I' \) is independent. Let \( r(A) \) be the rank of \( A \) in the binary field, i.e. the maximum cardinality of an independent subset of \( I \).

THEOREM 2.1 [6]

\[
 M(I) \leq r(A). 
\]

A solution basis is a set \( I' \subseteq I \) which corresponds to a maximal set of independent rows of \( A \). The cardinality of each solution basis is equal to \( r(A) \). Define the value of a solution basis \( I' \) as \( \sum_{i \in I'} c_i \). A maximum solution basis is a solution basis with maximum value.