REPRESENTATION THEOREM AND ONE-ITERATION THEOREM FOR FREDHOLM INTEGRAL EQUATION OF THE FIRST KIND $Ax = y$

Yun Tian-quan (云天铨)

(Department of Engineering Mechanics, South China University of Technology, Guangzhou)

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Abstract

In this paper, two theorems are presented. The representation theorem states: if the Fredholm integral equation of the first kind $Ax = y$, with bounded $L_2$ kernel, has a unique solution $\hat{x}$, then

$$\hat{x} = \sum_{n=1}^{\infty} (y, \varphi_n) \lambda_n \psi_n,$$

where $\varphi_n = \lambda_n A \psi_n$, $\psi_n = \lambda_n A^* \varphi_n$.

The one-iteration theorem states: $\hat{x}$ can be achieved in one iteration by $x_0 + g_0 A^*(y - Ax_0)$ if one of the following conditions is satisfied.

1. $\|A^* u_0\| \geq 2/\|A \|$, $u_0 = x_0 - x_0$;
2. $u_0 = g_0 A^* u_0$, $u_0 = y - Ax_0$;
3. $g_0 = \|A^* u_0\|^2 / \|A^{*} A u_0\|^2 = \|u_0\|^2 / \|A^{*} u_0\|^2$, $u_0 = y - Ax_0$ or $g_0 = \|A u_0\|^2 / \|A^* A u_0\|^2 = \|u_0\|^2 / \|A^* u_0\|^2$, $u_0 = Ax_0 - x_0$.

I. Introduction

Many problems of science and engineering\cite{1}\cite{8} can be reduced to Fredholm integral equation of the first kind $Ax = y$. For example: the problems in optics\cite{1}, X-ray\cite{2}, meteorology\cite{3}, polymer chemistry\cite{4}, mechanics\cite{5}\cite{7}, etc. Although the existence theorem of Fredholm integral equation of the first kind was already established by E. Picard (1910)\cite{9}, this theorem neither shows the solution in an explicit form nor connects with the ways of getting a solution. In 1951, L. Landweber\cite{10} gave an iteration formula $x_{n+1} = x_n + c A^* (y - Ax_n)$, where $0 < c$ (constant) $< 2/\|A^* A\|$, for getting a solution. Later, in 1970, Diaz and Metcalf\cite{11} discussed the relation between the Picard’s criterion and the convergence of the above sequence $\{x_n\}$. In 1978, the author\cite{12} proved the above sequence $\{x_n\}$ strongly convergent by Banach Contraction Mapping Theorem and suggested an iteration formula $x_{n+1} = x_n + g_n A^* (y - Ax_n)$ with fastest convergence, where the above constant $c$ unchanged in the whole iteration process is replaced by a constant $g_n = \|u_n\|^2 / \|A^* u_n\|^2$, $u_n = y - Ax_n$ varying for each iteration. Soon after, this formula was used in the calculation of predicting the thrust deduction of a propeller behind a body of revolution by J.L. Yuan (China Ship Scientific Research Center)\cite{13} and the calculating results with good agreement with experiments showed that the convergence was obtained by only one iteration by this formula for various cases while 3 to 5 iterations by Landweber’s formula for the same accuracy. It would be meaningful that if the solution of the Fredholm integral equation of the first kind $Ax = y$ can be expressed in an explicit form or it can be achieved by one iteration. In this paper, I shall present an explicit form of the solution $\hat{x}$ and study the conditions of getting $\hat{x}$ in one iteration by the fastest iteration formula.

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II. Representation Theorem

We consider a linear operator $A(\neq 0)$ defined by

$$Ax = \int_{\Omega} K(s, t)x(t)dt$$

with non-null bounded $L_2$ kernel $K(s, t)$ on $\Omega \times \Omega$ ($\Omega = [a, b]$), of a real Hilbert space $H$. Let $R_A$, $R_A^*$ and $N_A, N_A^*$ be the range and null space of $A$ and $A^*$ (adjoint of $A$) respectively. We have:

**Lemma**

$$A^*u \in L_2(\Omega) \cap R_A^* \quad \text{for } u \in L_2(\Omega) \cap R_A$$

$$Av \in L_2(\Omega) \cap R_A \quad \text{for } v \in L_2(\Omega) \cap R_A^*$$

**Proof** By Schwarz inequality, we have

$$\|A^*u\|^2 = |(A^*u, A^*u)| = |(u, AA^*u)| \leq \|u\| \cdot \|AA^*u\|$$

Since $u \in L_2(\Omega) \cap R_A$, so $\|u\| > 0$ and $u \in R_A = N_A^*$, i.e.,

$$u \notin N_A^* = \{w | A^*w = 0\}, \text{ or } A^*u \neq 0$$  \hspace{1cm} (2.5)

Thus (2.4) shows that

$$\|AA^*u\| \geq \|A^*u\|^2/\|u\| > 0, \quad \text{i.e., } AA^*u \neq 0, \text{ or } A^*u \notin N_A = R_A^*, \text{ i.e., } A^*u \in R_A^*$$  \hspace{1cm} (2.6)

Let $|K(s, t)| \leq M < \infty$. Then $\|A^*u\| \leq M \|u\| < \infty$, so we have (2.2). Similarly, we have (2.3).

Combining (2.2) and (2.3), we have:

$$A^*: L_2(\Omega) \cap R_A^* \rightarrow L_2(\Omega) \cap R_A$$

**Representation Theorem**

Suppose that the Fredholm integral equation of first kind

$$\int_{\Omega} K(s, t)x(t)dt = y(s), \quad y(s)(\neq 0) \in L_2(\Omega)$$

with bounded $L_2$ kernel $K(s, t)$ defined on $\Omega \times \Omega$ ($\Omega = [a, b]$) has a unique solution $x$, then $x \in L_2(\Omega)$ can be expressed in an explicit form

$$x = \sum_{n=1}^{\infty} (y, \varphi_n) \lambda_n \psi_n$$

where $[\varphi_n, \psi_n]$ is a pair of singular functions of $K$ belonging to the singular value $\lambda_n$, i.e.,

$$\varphi_n = \lambda_n A\psi_n = \lambda_n^2 A^*A\varphi_n, \quad \psi_n = \lambda_n A^*\varphi_n = \lambda_n^2 A^*A^*\psi_n$$

**Proof** Since $y \in L_2(\Omega)$ and $K(s, t)$ is a bounded $L_2$ kernel, i.e.,

$$\int_{\Omega} |K(s, t)|^2dt, \quad \int_{\Omega} |K(s, t)|^2ds, \quad \int_{\Omega} \int_{\Omega} |K(s, t)|^2dsdt$$

are bounded, therefore, by Schmidt's theorem [9, p. 160], we have:

1) "\*" means equal almost everywhere.