HAMILTONIAN DESCRIPTION OF BAROTROPIC ROSSBY WAVES

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A transformation into the normal canonical variables is found in the beta-plane approximation for barotropic Rossby waves of an arbitrary amplitude. This transformation is used to derive a matrix of three-wave interaction and to find an expression for the fourth-order term in the interaction Hamiltonian, which describes the modulation instability of Rossby waves. An increment of this instability has been calculated and estimated numerically.

1. INTRODUCTION

In the general case, along with the description of canonical variables of the system, the Hamiltonian theory of nonlinear Rossby waves incorporates the law of canonical transformation of the above variables into normal variables of the problem. The search for canonical variables for Rossby waves is discussed in many papers [1-4]. However, the normal canonical variables were obtained only for the particular case of small-amplitude Rossby waves [1], which rules out the solution in the form of solitons or localized vertices.

In this paper, the transformation into normal canonical variables, in which no limitation is imposed on the interacting-wave amplitude, is found for barotropic Rossby waves in the beta-plane approximation. The method we use is based on the variational principle [5-6] and is close to that used to study the Hamiltonian structure of Rossby waves in [3-4]. Using the above transformation, we obtain an expression for the three-wave interaction matrix. We also found an expression for a fourth-order term in the interaction Hamiltonian, which describes the modulation instability of Rossby waves. An increment of this instability is obtained and estimated numerically.

2. NORMAL CANONICAL VARIABLES

Let us consider the nonlinear Rossby waves in the beta-plane approximation. The x axis of the coordinate system is directed to the east, while the y axis is directed to the north, and \( \Omega \) is the angular velocity of the planet rotation. In this approximation, the equations of incompressible-liquid dynamics have the form

\[
\frac{du}{dt} - 2\Omega v \sin \lambda = \frac{1}{\rho_0} \frac{\partial p}{\partial x},
\]

\[
\frac{dv}{dt} + 2\Omega \sin \lambda u = -\frac{1}{\rho_0} \frac{\partial p}{\partial y},
\]

\[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0,
\]

where \( \vec{v} = (u, v) \), and \( \sin \lambda(y) \) is the latitude dependence of the Coriolis parameter. In the linear approximation it is written in the form

\[
\sin \lambda(y) = \sin \lambda_0 + \frac{\gamma_0}{2\Omega} y.
\]

Here \( \gamma_0 = \frac{2\lambda}{r_0} \cos \lambda_0 \), where \( r_0 \) is the Earth's radius. Since in the discontinuity equation (2) only one velocity
component remains independent, and the pressure \( p \) is determined via the velocity, Eq. (1) describes a system with one degree of freedom, and, hence, we need to find only one couple of canonical variables for a complete Hamiltonian description of the dynamics of the system studied. To find the canonical variables of the system considered, let us use the Clebsch transformation \([5]\) by writing the velocity projections \( u \) and \( v \) on the coordinate-system axis in the form

\[
\begin{align*}
\frac{\partial \varphi}{\partial x} + \alpha \frac{\partial \beta}{\partial x} - \sqrt{2\Omega} \alpha, \\
\frac{\partial \varphi}{\partial y} + \alpha \frac{\partial \beta}{\partial y} - \sqrt{2\Omega} \sin \beta.
\end{align*}
\]

In the Clebsch formulation (3), the equations of motion are obtained by the variational principle

\[
\delta \int dt \int \mathcal{L} \, dz \, dy = 0,
\]

where \( \mathcal{L} = -\rho_0 \left( \dot{\varphi} + \alpha \dot{\beta} + \frac{1}{2} u^2 + \frac{1}{2} v^2 \right) \) is the Lagrangian density.

Knowing the Lagrangian density \( \mathcal{L} \) and using the standard Legendre transformation, we obtain an expression for the Hamiltonian density \( \mathcal{H} \)

\[
\mathcal{H} = \frac{\partial \mathcal{L}}{\partial \dot{\varphi}} \dot{\varphi} + \frac{\partial \mathcal{L}}{\partial \dot{\beta}} \dot{\beta} - \mathcal{L},
\]

where the variables \( \beta \) and \( \partial \mathcal{L} / \partial \dot{\beta} = -\rho_0 \alpha \) can be considered as canonical conjugates of the system coordinate and momentum (see, for example, [3]), for which the initial equations of motion

\[
\begin{align*}
\frac{d\beta}{dt} - \sqrt{2\Omega} u &= 0, \\
\frac{d\alpha}{dt} + \sqrt{2\Omega} v \sin \gamma &= 0,
\end{align*}
\]

as is easily seen, are obtained from the Hamilton equations

\[
\dot{\beta} = \frac{\delta H}{\delta (-\rho_0 \alpha)}, \quad -\rho_0 \dot{\alpha} = -\frac{\delta H}{\delta \beta},
\]

in which the Hamiltonian \( H = \int \mathcal{H} \, dz \, dy \) is equal to the total energy of the medium

\[
H = \frac{\rho_0}{2} \int (u^2 + v^2) \, dz \, dy.
\]

We note that Eqs. (4) match Eqs. (1) after the substitution of Eq. (3) and allowance for the continuity equation (2). In other words, the mathematical description of the dynamics of the system under consideration, i.e., the system with one degree of freedom, is confined to Eqs. (4), and, therefore, the obtained canonical variables \( \beta \) and \( -\rho_0 \alpha \) completely solve the problem of the Hamiltonian description of this dynamics.

Let us move from the variables \( \beta \) and \( -\rho_0 \alpha \) to normal canonical variables which allow us to describe the nonlinear interaction of Rossby waves. For this purpose, we represent the initial variables of the problem in the form of the Fourier integral

\[
\begin{bmatrix}
\begin{array}{c}
\mathcal{u} \\
\mathcal{v} \\
\mathcal{\beta} \\
-\rho_0 \alpha \\
\varphi
\end{array}
\end{bmatrix} = \int_{k_x < 0} \begin{bmatrix}
\mathcal{u}_k \\
\mathcal{v}_k \\
\beta_k \\
-\rho_0 \alpha_k \\
\varphi_k
\end{bmatrix} e^{ik_x x + ik_y y} \, dk_e \, dk_y
\]