CONVEXLY INDEPENDENT SETS

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A family of pairwise disjoint compact convex sets is called convexly independent, if none of its members is contained in the convex hull of the union of the other members of the family. The main result of the paper gives an upper bound for the maximum cardinality \( h(k, n) \) of a family \( \mathcal{F} \) of mutually disjoint compact convex sets such that any subfamily of at most \( k \) members of \( \mathcal{F} \) is convexly independent, but no subfamily of size \( n \) is.

By an oval we mean a compact convex set in the plane. Let \( \mathcal{F} \) be a family of mutually disjoint ovals. A member \( A \) of \( \mathcal{F} \) is called a vertex of \( \mathcal{F} \) if

\[
A \not\in \text{conv}(\bigcup_{X \in \mathcal{F} \setminus \{A\}} X),
\]

otherwise \( A \) is called an internal member of \( \mathcal{F} \). The family \( \mathcal{F} \) is called convexly independent, if all of its members are vertices.

In [1] we proved that for any integer \( n \geq 4 \), there is a smallest natural number \( g(n) \) such that if \( \mathcal{F} \) is a family of more than \( g(n) \) mutually disjoint ovals with the property that any three members of \( \mathcal{F} \) are convexly independent, then \( \mathcal{F} \) contains \( n \) convexly independent members. This result generalizes a well-known theorem of Erdős and Szekeres [3] which concerns the case when all members of \( \mathcal{F} \) are points. In [1] and [2], we proved that \( g(4) = 4 \) and \( g(5) = 8 \); however, for arbitrary \( n \), we were able to give only a very poor upper bound for \( g(n) \). As a matter of fact, we did not give an explicit upper bound for \( g(n) \), but following the main steps of the proof in [1] and using known upper bounds for the Ramsey numbers one readily obtains that

\[
g(n) \leq t_n \left( t_{n-1} \left( \ldots (t_1 (c_n n)) \ldots \right) \right),
\]

where \( t_n(x) \) is the \( n \)-th tower function defined by \( t_1(x) = x \) and

\[
t_{n+1}(x) = 2^{t_n(x)}.
\]

In this article we shall consider families \( \mathcal{F} \) of mutually disjoint ovals with the property that, for a fixed integer \( k \geq 3 \), any \( m \leq k \) members of \( \mathcal{F} \) are convexly independent. It will be convenient to refer to this property as property \( H_k \). Further,
we say that $\mathcal{F}$ satisfies property $H^n$ if no $n$ members of $\mathcal{F}$ are convexly independent, and $\mathcal{F}$ has property $H^n_k$ if it satisfies both $H_k^n$ and $H^n$. With these notions our theorem mentioned above states that the cardinality of a family of mutually disjoint ovals satisfying property $H^n_k$ is bounded. It is natural to ask: What is the maximum cardinality $h(k, n)$ of a family of mutually disjoint ovals satisfying property $H^n_k$ for given $k$ and $n$, $3 \leq k \leq n$? The existence of the numbers $h(k, n)$ is guaranteed by the fact that property $H^n_k$ implies $H^n_\ell$ for $\ell < k$. Thus we have

$$h(k, n) \leq h(3, n) = g(n).$$

Of course, one can expect that, for $k > 3$, $h(k, n)$ is considerably smaller than $h(3, n)$. Our main result is an upper bound for $h(k, n)$ which is of the magnitude $c^n$ for $k = 4$ and of the magnitude $cn^2$ for $k \geq 5$.

**Theorem.** We have

$$h(4, n) \leq (n - 4) \left( \frac{2n - 4}{n - 2} \right) - n + 7$$

and

$$h(k, n) \leq (n - 3) \left\lceil \frac{n - 4}{k - 4} \right\rceil + n - 1$$

for $5 \leq k \leq n$.

The following example shows that

$$n - 1 + \left\lceil \frac{n - 1}{k - 2} \right\rceil \leq h(k, n).$$

Let $a_1, \ldots, a_{n-1}$ be the vertices of a convex $n - 1$-gon. For $i = 1, 2, \ldots, n - 1$ let $r_i$ be the point of intersection of the diagonals $a_{i-1}a_{i+1}$ and $a_ia_{i+2}$. Here, and throughout the construction we consider all indices modulo $n - 1$. Let $p_i$ and $q_i$ for $i = 1, \ldots, n - 1$ be the bisecting points of the segments $a_ir_i$ and $r_ia_{i+1}$ respectively. For $\ell = n, n + 1, \ldots, m = n - 1 + \left\lceil \frac{n - 1}{k - 2} \right\rceil$ we define the set $A_\ell$ as

$$A_\ell = \text{conv} \left\{ q_{\ell(k-2)-k+3}, p_{\ell(k-2)-k+4}, \ldots, q_{\ell(k-2)}, p_{\ell(k-2)+1} \right\}.$$

Then it is easy to check that the family $\mathcal{F}$ consisting of the ovals $A_1 = a_1$, $A_2 = a_2, \ldots, A_{n-1} = a_{n-1}$, $A_n, \ldots, A_m$ satisfies property $H^n_k$. Generally, we can add further members to the family obtained in this way without violating property $H^n_k$. In Fig. 1 the case $n = 13$, $k = 4$ is depicted. Here three further ovals can be inserted, and property $H^n_{13}$ still holds. However, using this “greedy” construction, we cannot obtain a family with property $H^n_k$ whose cardinality exceeds $cn$.

Our theorem, together with Ramsey’s theorem, yields a considerable improvement upon our previous upper bound for $g(n) = h(3, n)$. We recall that, for positive integers $k$, $\ell_1$ and $\ell_2$, the Ramsey number $R_k(\ell_1, \ell_2)$ is the minimal integer $m$ with the property that if the $k$-element subsets of a set $S$ of cardinality $m$ are coloured with two colours, red and blue, say, then there is a subset $S_1$ of $S$ with cardinality $\ell_1$, such that all $k$ element subsets of $S_1$ are red, or there is a subset $S_2$ of $S$ with