The purpose of this paper is to show that all Gödel's many valued propositional calculi and Dummett's linear calculus are structurally complete.

Gödel's calculi $G_n$ are given by matrices $\mathcal{M}_n$ of Gödel. The sequence of matrices $\mathcal{M}_n$ was introduced by Gödel in [2] and was axiomatized by Thomas in [5]. Hence $G_n$ are called Thomas's calculi $LC_n$ as well.

Dummett's linear calculus $LC$ was studied in [1] by Dummett. These calculi belong to the class of intermediate or superconstructive (superintuitionistic) propositional calculi. A superconstructive propositional calculus is formed from the intuitionistic propositional calculus $H$ by the addition of finite number of extra axioms.

1. Let $At$ be the set of all propositional variables $p_1, p_2, \ldots$, let $S$ be the set of well-formed formulas built by means of variables $p_1, p_2, \ldots$ and connectives: $\neg$ (negation), $\rightarrow$ (implication), $\wedge$ (conjunction), $\vee$ (disjunction). The variables $\alpha, \beta, \ldots$ run over the set $S$.

If $R$ is a nonempty, finite set of $n$-ary ($n > 1$) rules $r^n$ (resp. $r$), where $r^n \subseteq S^n$, and $A \subseteq S$, then the couple $\langle R, A \rangle$ is called the system of propositional calculus with axioms $A$ and primitive rules $R$. The rule $r^n$ is structural, $r^n \in \text{Struct}$, iff for every $e : At \rightarrow S$ and for every $\alpha_1, \ldots, \alpha_n \in S$: if $r^n(\alpha_1, \ldots, \alpha_n)$, then $r^n(h^e(\alpha_1), \ldots, h^e(\alpha_n))$, where $h^e$ is the extension of $e$ to an endomorphism $h^e : S \rightarrow S$. The set $\text{Cn}(R, X)$ is the least set containing $X$ and closed under each of the rules $r \in R$. The rule $r$ is permissible in $\langle R, A \rangle$, $r \in \text{Perm}(R, A)$, iff $\text{Cn}(R \cup \{r\}, A) \subseteq \text{Cn}(R, A)$; the rule $r$ is derivable in $\langle R, A \rangle$, $r \in \text{Der}(R, A)$, iff $\text{Cn}(R \cup \{r\}, A \cup X) \subseteq \text{Cn}(R, A \cup X)$, for every $X \subseteq S$.

The symbol $r_0$ denotes the modus ponens rule and the symbol $r_*$ denotes the substitution rule, the sets $R_0$ and $R_{0*}$ are defined by the equations $R_0 = \{r_0\}$, $R_{0*} = \{r_0, r_*\}$ respectively. $\text{Sb}(X)$ is the smallest set containing $X \subseteq S$ and closed under the substitution rule.

2. The system $\langle R, A \rangle$ is structurally complete, i.e. $\langle R, A \rangle \in \text{SCpl}$ iff $\text{Struct} \cap \text{Perm}(R, A) \subseteq \text{Der}(R, A)$ (the notion of structural completeness is introduced by W. A. Pogorzelski in [3]).

T. Prucnal [4] proved that the system $\langle R, A \rangle$ is structurally complete iff for every finite set $\pi \subseteq S$ and for every $\beta \in S$:

\[ (*) \forall e : At \rightarrow S \quad (h^e(\pi) \subseteq \text{Cn}(R, A) \Rightarrow h^e(\beta) \in \text{Cn}(R, A)) \Rightarrow \beta \in \text{Cn}(R, A \cup \pi). \]
Classical propositional calculus \( \langle R_\omega, Sb (A_\omega) \rangle \) is structurally complete [3]. This calculus is the second element of sequence of calculi \( G_n \) and these calculi begining with the third one are, as mentioned above, intermediate calculi between classical and intuitionistic. The latter one is known as lacking structural completeness (A. Wroński, T. Prucnal). Hence there arises a problem of structural completeness of Gödel’s and Dummett’s calculi. (W. A. Pogorzelski proved that \( \langle R_{0\omega}, A_{0\omega} \rangle \in \text{Scpl} \), where \( A_{0\omega} \) is set of axioms of Gödel’s matrix \( M_0 \)). Till now there is already solved the problem of structural completeness of many valued Łukasiewicz’s calculi, Lewis’s systems S4, S5 and others.

3. By \( n \)-th Gödel’s matrix \( M_n, n \in N, \) (cf. [2]), we mean an algebra \( \langle |M_n|, \Omega_n \rangle \) with designated value \( \{1\} \) i.e. \( M_n = \langle |M_n|, \Omega_n, \{1\} \rangle \), where \( |M_n| = \{1, 2, \ldots, n\} \), \( \Omega_n = \{f^*_n, f^-_n, f^-_n, f^*_n\} \) and for every \( x, y \in |M_n| \)

\[
\begin{align*}
  f^*_n(x) &= \begin{cases} 
  n, & \text{if } x < n, \\
  1, & \text{if } x = n,
  \end{cases} \\
  f^-_n(x, y) &= \begin{cases} 
  1, & \text{if } x \geq y, \\
  y, & \text{if } x < y,
  \end{cases} \\
  f^-_n(x, y) &= \max(x, y), \\
  f^*_n(x, y) &= \min(x, y).
\end{align*}
\]

(The set of all positive integers is denoted by \( N \)). The set of all formulas valid in matrix \( M_n \) will be denoted by \( G_n \), i.e. \( \alpha \in G_n \) iff \( v(\alpha) = 1 \), for every valuation \( v : S \rightarrow |M_n| \). In [5] Thomas proved that \( G_n = \text{Cn}(R_{0\omega}, A_{0\omega}), n \in N \), where \( A_{0\omega} = H \cup \bigcup \{ T_n \} \) and \( T_n \) is defined: \( T_1 = p_1, T_{n+1} = ((p_n \rightarrow p_{n+1}) \rightarrow p_1) \rightarrow T_n \).

By matrix \( M_\omega \) we mean an algebra \( \langle |M_\omega|, \Omega_\omega \rangle \) with designated value \( \{1\} \), i.e. \( M_\omega = \langle |M_\omega|, \Omega_\omega, \{1\} \rangle \), where \( |M_\omega| = \{1, 2, \ldots, n, \ldots, \omega\} \), \( \Omega_\omega = \{f^*_\omega, f^-_\omega, f^-_\omega, f^*_\omega\} \) and for every \( x, y \in |M_\omega| \)

\[
\begin{align*}
  f^*_\omega(x) &= \begin{cases} 
  \omega, & \text{if } x < \omega, \\
  1, & \text{if } x = \omega,
  \end{cases} \\
  f^-_\omega(x, y) &= \begin{cases} 
  1, & \text{if } x \geq y, \\
  y, & \text{if } x < y,
  \end{cases} \\
  f^-_\omega(x, y) &= \max(x, y), \\
  f^*_\omega(x, y) &= \min(x, y), (\text{cf. [1]}).
\end{align*}
\]

The set of all formulas valid in the matrix \( M_\omega \) will be denoted by \( LC \), i.e. \( \alpha \in LC \) iff \( v(\alpha) = 1 \), for every valuation \( v : S \rightarrow |M_\omega| \). In [1] Dummett showed, that \( LC = \text{Cn}(R_\omega, A_{LC}) \) where \( A_{LC} = H \cup \{(p_1 \rightarrow p_2) \lor (p_2 \rightarrow p_3)\} \) and that

\[
(\ast \ast \ast) \quad G_n = \bigcap_{n \in N} G_n.
\]

From the definition given above it is easy to see that

\[
\text{(\ast \ast \ast \ast)} \quad H \subset LC \subset \ldots \subset G_{n+1} \subset G_n \subset \ldots \subset G_2 \subset G_1 = S
\]

where \( X \subset Y \) means that \( X \subseteq Y \) and \( X \neq Y \).

**Lemma 1.** If \( \alpha \in G_{n-1} \subset G_n \); then there exists a valuation \( v : S \rightarrow |M_n| \) such that \( v(\alpha) = 2 \), \( n \in N \).

**Proof.** We will show that if \( \alpha \in G_{n-1} \subset G_n \) then \( v(\alpha) \leq 2 \), for every \( v : S \rightarrow |M_n| \). It is easy to see that the mapping \( h : |M_n| \rightarrow |M_{n-1}| \) defined:

\[
h(1) = h(2) = 1; \quad h(k) = k-1, \text{ for } k = 3, \ldots, n.
\]

is a homomorphism of an algebra \( \langle |M_n|, \Omega_n \rangle \) onto an algebra \( \langle |M_{n-1}|, \Omega_{n-1} \rangle \).