ON THE PROBABILITY PROPERTIES OF THE DENSITY
GRADIENT OF A RANDOMLY MOVING
INCOMPRESSIBLE MEDIUM

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We study the statistical properties of the density-field gradient of a passive admixture in a randomly moving medium. A expression for the relative gradient-variation field is derived. It is shown that the moments and practically all realizations of the relative gradient-variation field increase exponentially with time.

1. INTRODUCTION

The evolution of the random field of a passive admixture in chaotically moving media attracts the increased attention of researchers (see, for example, [1–3]). This is explained by the fact that knowledge of the statistical properties of passive admixtures is important in the solution of various problems, for example, in atmosphere and ocean pollution analysis. In this case, the majority of studies of passive-admixture transfer and diffusion are usually confined to attempts to derive equations for the average admixture density or diffusion analysis of an individual particle of the admixture [4–8]. However, the above properties cannot describe many important features of passive-admixture density fields, such as, the appearance of a complex layered structure of the passive admixture whose density changes abruptly with slight displacements of the observation point. A more adequate description of this feature of passive-admixture fields, among many others, is given by their probability properties, which are analyzed in this paper disregarding the molecular-diffusion influence.

2. STATISTICAL DESCRIPTION OF DENSITY GRADIENT IN DIFFUSION APPROXIMATION

Although in an incompressible medium we have neither compression nor tension, which are responsible for density fluctuations in a compressible medium, in incompressible chaotic motion the initially smooth density profiles become more and more irregular, because of the fact that admixture particles that are initially close to each other disperse, while particles that are far from each other can get closer. As a result, parts of the medium with quite different admixture densities can be adjacent, leading to sharp density jumps at close points. Let us try to represent a qualitative description of such increasing spatial irregularity of the density fields in a chaotically moving incompressible medium. For simplicity we confine ourselves to a 2D case in which the velocity-field components are expressed via the current function $\psi(\mathbf{x}, t)$ in the following manner:

$$
\begin{align*}
  v_1 &= \frac{\partial \psi}{\partial z_2}, \\
  v_2 &= -\frac{\partial \psi}{\partial z_1},
\end{align*}
$$

the statistical properties of which are assumed to be known. The irregularity of density-field realizations in space is quantitatively characterized, for example, by the density-field gradient $\nabla_x \rho(\mathbf{x}, t)$. Let us study its statistical properties. For this purpose we recall that the density field in an incompressible medium is given by the expression

$$
\rho(\mathbf{x}, t) = \rho_0(\hat{\mathbf{Y}}(\mathbf{x}, t)),
$$

where $\rho_0(\vec{x})$ is the initial determinate density profile, and $\vec{y} = \vec{Y}(\vec{x}, t)$ is the law of transformation of Euler coordinates into Lagrange coordinates. As a result, we obtain

$$\vec{\nabla}_{x}\rho = \vec{e}_1 \left( \frac{\partial \rho_0}{\partial y_1} j_{11} + \frac{\partial \rho_0}{\partial y_2} j_{12} \right) + \vec{e}_2 \left( \frac{\partial \rho_0}{\partial y_1} j_{21} + \frac{\partial \rho_0}{\partial y_2} j_{22} \right).$$

(3)

Here $\vec{e}_1$, and $\vec{e}_2$ are unit vectors of the Euler coordinate system and $j_{lm}$ are components of the tensor of transfer from Lagrange to Euler coordinates

$$j_{lm} = \frac{\partial Y_l(\vec{y}, t)}{\partial x_m}, \quad l, m = 1, 2.$$  

(4)

The statistical properties of gradient (3) are most conveniently studied in the Lagrange coordinate system. Therefore, we represent $j_{lm}(\vec{z}, t)$ in Eq. (3) via the components of the tensor of transfer from Euler to Lagrange coordinates

$$J_{lm}(\vec{y}, t) = \frac{\partial X_1(\vec{y}, t)}{\partial y_m}.$$  

(5)

Differentiating for this purpose this obvious vector identity $\vec{z} = \vec{X}(\vec{Y}(\vec{x}, t), t)$ with respect to $z_1$ and $z_2$ and solving the resulting equations with respect to $j_{lm}$, we obtain

$$j_{11} = J_{22}/J, \quad j_{22} = J_{11}/J, \quad j_{12} = -J_{12}/J, \quad j_{21} = -J_{21}/J.$$  

(6)

Substituting Eq. (6) into Eq. (3) and taking into account that in the case of an incompressible medium $J = 1$, we obtain

$$\vec{\nabla}_{x}\rho = |\vec{\nabla}_x\rho_0|(\vec{e}_1 J_1 + \vec{e}_2 J_2).$$  

(7)

where $|\vec{\nabla}_x\rho_0|$ is the initial absolute value of the density gradient at point $\vec{y}$,

$$J_1(\vec{y}, t, \theta_0) = J_{22} \cos \theta_0 - J_{21} \sin \theta_0,$$

$$J_2(\vec{y}, t, \theta_0) = J_{11} \sin \theta_0 - J_{12} \cos \theta_0,$$

(8)

and $\theta_0$ is the angle between the $y_1$ axis and the initial direction of the gradient. It is easily shown that in the Lagrange coordinate system the random fields $J_1$ and $J_2$ (8) satisfy the following system of stochastic equations

$$\begin{cases}
\frac{dJ_2}{dt} = \delta J_2 - \beta J_1, & J_1(\vec{y}, t = 0, \theta_0) = \cos \theta_0, \\
\frac{dJ_1}{dt} = -\delta J_2 + \gamma J_2, & J_2(\vec{y}, t = 0, \theta_0) = \sin \theta_0.
\end{cases}$$  

(9)

Here we use the notation

$$\delta(\vec{z}, t) = \frac{\partial^2 \psi(\vec{z}, t)}{\partial x_1 \partial x_2}, \quad \beta(\vec{z}, t) = \frac{\partial^2 \psi(\vec{z}, t)}{\partial x_2^2}, \quad \gamma(\vec{z}, t) = \frac{\partial^2 \psi(\vec{z}, t)}{\partial x_1^2}.$$  

(9')

In turn, we present the solutions of Eq. (9) in the form

$$J_1 = e^\delta \cos \theta, \quad J_2 = e^\delta \sin \theta.$$  

(10)

Then the expression for the gradient (7) becomes especially obvious

$$\vec{\nabla}_{x}\rho = |\vec{\nabla}_x\rho_0| g(\vec{y}),$$  

(11)

where

$$g(\vec{y}, t, \theta_0) = e^\delta = \frac{|\vec{\nabla}_{x}\rho|}{|\vec{\nabla}_x\rho_0|}.$$  

(12)