On the Algebraization of a Feferman’s Predicate

(The algebraization of theories which express Theor; X)

Summary. This paper is devoted to the algebraization of an arithmetical predicate introduced by S. Feferman. To this purpose we investigate the equational class of Boolean algebras enriched with an operation \( g \), which translates such predicate, and an operation \( \tau \), which translates the usual predicate Theor. We deduce from the identities of this equational class some properties of \( g \) and some ties between \( g \) and \( \tau \); among these properties, let us point out a fixed-point theorem for a sufficiently large class of \( g-\tau \) polynomials. The last part of this paper concerns the duality theory for \( g-\tau \) algebras.

Introduction

Recently, R. Magari and other authors have studied how some meta-theorems of Peano arithmetic \( \mathcal{P} \) (and in general of theories satisfying some derivability conditions) can be expressed in algebraic terms. It has been emphasized that many results depend only on three properties of the predicate Theor, which can be written as identities of the Lindenbaum algebra of \( \mathcal{P} \) enriched with a unary operation \( \tau \) expressing Theor. Among these results, let us point out the two Gödel’s Theorems, Löb’s condition, the existence and uniqueness (up to provable equivalence) of the fixed-point for formulas built (in a reasonable sense) from variables, Theor, and Boolean connectives, also in the intuitionistic logic. The above-mentioned papers provide us with relatively simple techniques in the part of the proof-theory which is related to the Gödel’s theorems.

In this paper, we study, with the same techniques, the algebraization of a predicate which has been introduced by S. Feferman in [3], and is similar, in a sense, to Rosser’s predicate. Namely, we define, in the Lindenbaum sentence algebra of \( \mathcal{P} \), a unary operation \( q \), which can be regarded as an algebraic translation of Feferman’s predicate. Then, in paragraph 2, we study the equational class of Boolean algebras enriched with such an operator \( q \) (the algebras of this class will be called \( q \)-algebras); that allows us to obtain an algebraic counterpart of Rosser’s theorem. In paragraph 3 we consider the relations expressed by identities between the operations \( g \) and \( \tau \) in the Lindenbaum sentence algebra of \( \mathcal{P} \); these identities define the equational class of \( g-\tau \) algebras. In paragraph 4 we discuss both the problem of introducing an operator with the property of \( g \) into a diagonalizable algebra, and that of introducing an operation with the properties of \( \tau \) into a \( g \)-algebra. In this way we translate the logic problem

\[ \mathcal{P}(\overline{\mathcal{P}}) \rightarrow \mathcal{P}(\overline{T(\overline{\mathcal{P}})}) \] for every sentence \( p \) of \( \mathcal{P} \).
of building a "Theor predicate" starting from a "Rosser predicate" (see
the sequel for the definitions) and the inverse problem. Then (n. 5), we
prove a fixed-point theorem in every $\varphi$-p algebra for a sufficiently large
class of $\varphi$-p polynomials. Finally, we characterize the dual space of a Boolean
algebra enriched with the operations $\varphi$ and $\tau$. With regard to the last
problem, we recall that $\tilde{\varphi} = \forall \varphi \forall$ and $\sigma = \forall \tau \forall$ are hemimorphisms in the
sense of P. R. Halmos, [5], and hence they are associated with Boolean
relations defined in the dual space.

For the sake of simplicity, we always refer to Peano arithmetic $\mathcal{P}$,
although there exists a large class of theories for which the same results
can be obtained: for instance, all these theories in which $\mathcal{P}$ is relatively
interpretable.

1. Preliminary notes

We say that a formula $\hat{T}(x)$ of $\mathcal{P}$, with just one free variable $x$, is a
**Theor predicate** if the set $\{p: \vdash_{\mathcal{P}} \hat{T}(\overline{p})\}$ is exactly the set of theorems of $\mathcal{P}$, and
if, for every two propositions $p$, $q$ of $\mathcal{P}$, we have: a) $\vdash_{\mathcal{P}} [\hat{T}(\overline{p}) \land \hat{T}(\overline{p \rightarrow q})]$
$\rightarrow \hat{T}(\overline{q})$ and b) $\vdash_{\mathcal{P}} \hat{T}(\overline{p}) \rightarrow \hat{T}(\overline{T(p)})$. From a) and b), by diagonalization
Lemma, we obtain both L"ob's theorem and its formalization: $\vdash_{\mathcal{P}} \hat{T}(\overline{T(\overline{\varphi(p)}) \rightarrow \overline{\varphi(p)}}) \rightarrow \hat{T}(\overline{\varphi(p)})$. Moreover, let us note that, as a particular case of L"ob's theo-
rem, we have the second theorem of G"odel, that is: $\not \vdash_{\mathcal{P}} \text{Con}_T$, where
$\text{Con}_T$ is the sentence $\forall x \forall x \exists \hat{T}(x) \land \hat{T}(\overline{\varphi(x)})$.

On the other hand, we shall say that $\hat{R}(x)$ is a **Rosser predicate**, if the
set $\{p: \vdash_{\mathcal{P}} \hat{R}(\overline{p})\}$ is the set of theorems of $\mathcal{P}$, and $\vdash_{\mathcal{P}} \text{Con}_R$. In order to obtain
an algebraization of Rosser's theorem, we have to define in the Linden-
bau algebra of $\mathcal{P}$ an operation $\varphi$ associated with a Rosser predicate
$\hat{R}(x)$ as follows: $\varphi[\overline{p}] = \hat{R}(\overline{p})$, where, for every sentence of $\mathcal{P}$, $[\overline{p}]$ denotes
the equivalence class of $p$ with respect to provable equivalence. Since
the above definition is unambiguous, $\varphi[\overline{p}]$ must depend only on $[\overline{p}]$ and
not on $p$. In other words, a necessary condition is: if $\vdash_{\mathcal{P}} p \leftarrow q$, then $\vdash_{\mathcal{P}} \hat{R}(\overline{p}) \leftarrow \hat{R}(\overline{q})$. On the ground of this remark, it is useful to consider the Fe-
ferman's predicate $\hat{F}(x)$ (see [3]'), which can be informally defined in the
following way: let $\Pi(x)$ be a formula which bimmutes "in a natural
way" the set $A$ of axioms of $\mathcal{P}$, and let $\Pi^*(x)$ be the formula $\Pi(x) \land \forall y (y

\begin{footnote}
\text{We use terminology and notations of R. Magari [7], [8]. In particular, $+\ldots v$,}
denote respectively the operations of join, meet, complementation. With regard to
the representation of recursive and recursively enumerable relations in a theory, we
sometime refer to S. Feferman, [3]. For the sake of simplicity, we consider as G"odel-
numbering of the set of propositions of $\mathcal{P}$ a primitive recursive bijection between this
set and the set of natural numbers.

\text{In [3], this predicate is denoted by $Pr^M_{\Pi^*(x)}$.}
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