Galerkin methods for a semilinear parabolic problem with nonlocal boundary conditions

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Dedicated to Professor J. Crank on the occasion of his 80th birthday

We formulate and analyze a Crank-Nicolson finite element Galerkin method and an algebraically-linear extrapolated Crank-Nicolson method for the numerical solution of a semilinear parabolic problem with nonlocal boundary conditions. For each method, optimal error estimates are derived in the maximum norm.

Keywords: semilinear parabolic problem, nonlocal boundary conditions, finite element Galerkin method, Crank-Nicolson method, extrapolated Crank-Nicolson method, optimal error estimates.

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1. Introduction

Consider the semilinear parabolic equation

$$u_t - (a(x)u_x)_x = F(u, x, t), \quad 0 \leq x \leq 1, \quad 0 \leq t \leq T, \quad (1.1)$$

subject to the initial condition

$$u(x, 0) = u^0(x), \quad 0 \leq x \leq 1, \quad (1.2)$$

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and the nonlocal boundary conditions
\[ u(0, t) = \int_0^1 \alpha(x) u(x, t) \, dx + g_0(t), \quad u(1, t) = \int_0^1 \beta(x) u(x, t) \, dx + g_1(t), \]
\[ 0 \leq t \leq T. \tag{1.3} \]

We assume that the function \( \alpha \) is continuous and that there exist constants \( a_1 \) and \( a_2 \) such that \( 0 < a_1 \leq a(x) \leq a_2, \ 0 \leq x \leq 1 \). Also, \( a' \in L^2(0, 1) \) and the function \( F \) is Lipschitz continuous on compact sets. This problem arises in the quasi-static theory of thermoelasticity [1, 2]. The existence, uniqueness and some properties of solutions of problems of the form (1.1)–(1.3), even in several space variables, have been studied by Day [1, 2], Friedman [6] and Kawohl [7] under the assumption that
\[ \int_0^1 |\alpha(x)| \, dx \leq \mu < 1, \quad \int_0^1 |\beta(x)| \, dx \leq \mu < 1. \tag{1.4} \]

We assume that the solution \( u \) is sufficiently smooth and that
\[ \int_0^1 |\alpha(x)|^2 \, dx \leq \mu^2 < 1, \quad \int_0^1 |\beta(x)|^2 \, dx \leq \mu^2 < 1. \tag{1.5} \]

These conditions are sufficient to ensure the existence and uniqueness of the solution of the problem (1.1)–(1.3) in the linear case [6], but are stronger than (1.4). We need these stronger conditions in order to use an energy argument.

Little appears in the literature on the numerical solution of problems of the form (1.1)–(1.3). For the linear case with \( a = 1 \) and \( F = F(x, t) \), Ekolin [5] proved the convergence of three finite difference methods – the forward Euler method, the backward Euler method and the Crank–Nicolson method. In these schemes, the integrals in the nonlocal boundary conditions are approximated by the trapezoidal rule. The convergence of the forward and backward Euler methods is proved under the assumption that (1.4) holds, whereas the analysis of the Crank–Nicolson method uses an energy argument which requires that
\[ \left( \int_0^1 |\alpha(x)|^2 \, dx \right)^{1/2} + \left( \int_0^1 |\beta(x)|^2 \, dx \right)^{1/2} \leq \sqrt{3}/2; \]

cf. (1.5). Lin et al. [8] studied semi-implicit and fully implicit backward Euler schemes for the two dimensional heat equation subject to a nonlocal condition which is an analogue of (1.3). They showed that the schemes preserve certain properties of the original problem, and presented some numerical results to support their theory.

The purpose of this paper is to formulate and analyze two discrete-time finite element Galerkin methods for the solution of (1.1)–(1.3). In order to describe these methods, we introduce the following notation. Let \( \Pi_h = \{x_j\}_{j=0}^J \) be a partition of the unit interval such that
\[ 0 = x_0 < x_1 < \cdots < x_J = 1. \]