Ordinals II:
Some applications and a functorial approach

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The first part of this paper is a presentation of some common applications of ordinals: definition of a system of ordinal notations for ordinals less than $\Gamma_0$, direct connection between Kruskal’s theorem and $\Gamma_0$, consistency proofs in proof theory (such as the consistency of Peano arithmetic by means of transfinite induction up to $\varepsilon_0$). In the second part of the paper, a functorial construction of ordinals and in particular of the Veblen hierarchy is explained. This approach, introduced by Girard (theory of dilators), allows the construction of ordinals greater than $\Gamma_0$ to be pursued in a more natural way than if the Bachmann hierarchy is used.

1. Introduction

The first part of this paper is a presentation of complements on ordinals or common applications of ordinals; section 2 goes on to define a system of ordinal notations for ordinals less than $\Gamma_0$. Section 3 concerns a simple direct connection between Kruskal’s theorem and $\Gamma_0$ [14]. It should be noted that Kruskal’s theorem plays an essential role in computer science since it can be used to show that certain orderings on trees, which are involved when proving the termination of certain rewriting systems, are indeed well-founded [5, 12]. In section 4, it is shown how ordinals can be used for consistency proofs in proof theory (such as the consistency of Peano arithmetic by means of transfinite induction up to $\varepsilon_0$).

In section 5, a functorial construction of ordinals and in particular of the Veblen hierarchy is introduced. This approach allows the construction of ordinals greater than $\Gamma_0$ to be pursued in a more natural way than if the Bachmann hierarchy is used [1]. The constructions are not given in detail, only the main tools. Further explanation can be found in [7, 9, 10, 15, 16].
2. A notation system for the ordinals \(<\Gamma_0\)

In this section, ordinal terms are constructed as formal strings of symbols for which a constructively defined \(<\)-relation exists. It is then possible to prove constructively all the theorems about ordinal terms: the proofs are easy but tedious, and the reader will find detailed proofs in [13]. This system of ordinal notations is used in section 3 to prove that Kruskal's theorem implies that transfinite induction holds for \(\Gamma_0\). Ordinal notations are also often used in proof theory.

2.1. COMPLEMENTS ON THE VEBLEN HIERARCHY

The following theorem gives the order between \(\mathcal{K}^{\alpha_1}(\beta_1)\) and \(\beta = \mathcal{K}^{\alpha_2}(\beta_2)\), and will be used in section 2.2.

**THEOREM 1**

Let \(\alpha = \mathcal{K}^{\alpha_1}(\beta_1)\) and \(\beta = \mathcal{K}^{\alpha_2}(\beta_2)\) be two ordinals:

(i) \(\alpha = \beta\) if and only if one of the following holds:

- \(\alpha_1 < \alpha_2\) and \(\beta_1 = \mathcal{K}^{\alpha_2}(\beta_2)\),
- \(\alpha_1 = \alpha_2\) and \(\beta_1 = \beta_2\),
- \(\alpha_2 < \alpha_1\) and \(\mathcal{K}^{\alpha_1}(\beta_1) = \beta_2\);

(ii) \(\alpha < \beta\) if and only if one of the following holds:

- \(\alpha_1 < \alpha_2\) and \(\beta_1 < \mathcal{K}^{\alpha_2}(\beta_2)\),
- \(\alpha_1 = \alpha_2\) and \(\beta_1 < \beta_2\),
- \(\alpha_2 < \alpha_1\) and \(\mathcal{K}^{\alpha_1}(\beta_1) < \beta_2\).

**Proof**

(i) If \(\alpha_1 < \alpha_2\), then \(\mathcal{K}^{\alpha_1}(\mathcal{K}^{\alpha_2}(\beta_2)) = \mathcal{K}^{\alpha_2}(\beta_2)\); therefore, since \(\mathcal{K}^{\alpha_1}\) preserves order, \(\alpha = \beta\) if and only if \(\beta_1 = \mathcal{K}^{\alpha_2}(\beta_2)\); the case of \(\alpha_2 < \alpha_1\) is similar; for \(\alpha_1 = \alpha_2\), since \(\mathcal{K}^{\alpha_1}\) preserves order, the assertion holds.

(ii) If \(\alpha_1 < \alpha_2\), then \(\mathcal{K}^{\alpha_1}(\mathcal{K}^{\alpha_2}(\beta_2)) = \mathcal{K}^{\alpha_2}(\beta_2)\); therefore, since \(\mathcal{K}^{\alpha_1}\) preserves order, \(\mathcal{K}^{\alpha_1}(\beta_1) < \mathcal{K}^{\alpha_2}(\beta_2)\) if and only if \(\beta_1 < \mathcal{K}^{\alpha_2}(\beta_2)\); the case of \(\alpha_2 < \alpha_1\) is similar; for \(\alpha_1 = \alpha_2\), since \(\mathcal{K}^{\alpha_1}\) preserves order, the assertion holds.

**DEFINITION 1**

- An ordinal \(\gamma\) is an **additive principal ordinal** if and only if \(\gamma \in \text{Im}(\mathcal{K}^0)\).
- An ordinal \(\gamma\) is **maximal \(\alpha\)-critical** if and only if \(\gamma \in \text{Im}(\mathcal{K}^\alpha)\) and \(\gamma \notin \text{Im}(\mathcal{K}^{\alpha+1})\).

Therefore, an ordinal \(\gamma\) is maximal \(\alpha\)-critical if and only if there exists \(\beta < \gamma\) such that \(\gamma = \mathcal{K}^\alpha(\beta)\); note that if \(\gamma\) is maximal \(\alpha\)-critical and \(\gamma = \mathcal{K}^\alpha(\beta)\), then \(\mathcal{K}^\alpha(\beta)\) is the generalized Cantor normal form of \(\gamma\).