SOMMARIO. Si caratterizza un solido termoelastico riguardando la seconda legge della termodinamica come una legge supplementare di conservazione compatibile con le equazioni del moto. Si ottengono delle condizioni sufficienti per l'iperbolicità del sistema base che può essere posto in forma simmetrica e conservativa.

SUMMARY. We characterize a thermoelastic solid by considering the entropy law as a supplementary conservation law consequence of the laws of motion. Sufficient conditions in order the governing equations to be a hyperbolic system are pointed out.

1. INTRODUCTION

As is well known, rational thermodynamic assumes the existence of entropy as an objective constitutive quantity. Moreover the second law is interpreted as a restriction to the constitutive response characterizing the medium. In other words [1] the second law may be interpreted as a supplementary conservation law which is a consequence of the laws of motion. In this paper we consider a thermoelastic solid within the framework of the thermodynamical theory proposed by I. Müller [2] because it may provide finite propagation speeds for thermal waves, the governing equations may be hyperbolic and the only present values of the field quantities are involved in the constitutive equations. All that allows one to study wave propagation problems easily. By using a procedure, which is a slight extension of that given in [3], [5] for any first order system of equation in conservative form to be endowed with a supplementary conservation law, we place restriction on the constitutive laws obtaining the same results as in [2]. Sufficient conditions in order the governing equations to form a hyperbolic system are pointed out. In such a case, by means of a suitable transformation of the field variables, it is possible to put the governing equations in the form of a symmetric and conservative first order quasilinear hyperbolic system for which the Cauchy problem results well posed.

2. THE SUPPLEMENTARY CONSERVATION LAW.

The balance laws, usually assumed to be valid in continuum thermodynamics are the conservation of mass, the balance of momentum and the conservation of energy.

Such equations, when we consider a supply free body (the body force and the supply density of energy vanishing), in material form are the following:

\[
\begin{align*}
\rho_0 \partial_t v_i - \partial X_A T_{iA} &= 0 \\
\rho_0 \partial_t \left( \frac{1}{2} v^2 + e \right) + \partial X_A (q_A - v_i T_{iA}) &= 0 \\
\rho_0 &= \rho I, \; J = \det \| \partial x_i / \partial X_A \|, \; \partial_t = \partial / \partial t, \; \partial X_A = \partial / \partial X_A
\end{align*}
\]

where \( t \) is time coordinate, \( x_i = x_i(X_A, t) \) and \( X_A \) are, respectively, the present position and the reference position, \( \rho \) is the density, \( e \) the specific internal energy, \( q_A \) is the heat flux, \( T_{iA} \) is the first Piola-Kirchoff stress tensor.

The previous equations provide, together with the entropy inequality, restrictions on the response of bodies once that some set of constitutive laws have been chosen. This suggest that the entropy law may be regarded as a supplementary conservation law which follows from the balance laws.

To be more precise we consider the following constitutive laws satisfying the objectivity principle [4]

\[
\begin{align*}
T_{AB} &= T_{AB}(C_{AB}, \theta, \dot{\theta}, \theta_A) \\
q_A &= q_A(C_{AB}, \theta, \dot{\theta}, \theta_A) \\
e &= e(C_{AB}, \theta, \dot{\theta}, \theta_A)
\end{align*}
\]

where

\[
\begin{align*}
C_{AB} &= F_{iA} F_{jB}, \quad T_{ij} = T_{AB} F_{iA} F_{jB} \\
\theta_A &= \partial X_A \theta, \quad \dot{\theta} = \partial_t \theta, \quad F_{iA} = \partial X_A x_i
\end{align*}
\]

\( \theta = \theta(X_A, t) \) is the empirical temperature and we assume, [2] that (2) characterize a simply thermoelastic body.

By using (2), the equations (1) may be written as a first order quasi-linear hyperbolic system is generalized conservative form:

\[
\partial_t V + \partial X_A f^A = f
\]

where

\((*)\) This work was partially supported by the C.N.R. (Gruppo Nazionale per la Fisica Matematica).

\((**)\) Dipartimento di Matematica, Università di Messina, Messina.
Now we require the system (3) to be endowed with a supplementary conservation law of the type

$$\rho_0 \partial_t h + \partial_{X_A} h_A = g \tag{4}$$

and, in order to satisfy the material indifference principle, we have

$$h = h(C_{AB}, \theta, \hat{\theta}, \theta_A) \quad h_A = h_A(C_{AB}, \theta, \hat{\theta}, \theta_A)$$

The condition (4) is equivalent, by making ad hoc assumption on $g$, to admit the entropy principle stated by I. Müller in [2].

The compatibility conditions of a first order quasi-linear hyperbolic system in conservative form with a supplementary conservation law (4), have been deduced in [3], [5] where, among other results, it is shown that it is possible to put the original system in symmetric form. Consequently, if the system is hyperbolic, the Cauchy problem results well posed when initial data are assumed smooth on a space like initial surface.

Such compatibility conditions are no longer valid in our case [1] because of the constraints:

$$\partial_{X_A} \theta = \theta_A; \quad \partial_{X_B} \theta_B = \partial_{X_B} \theta_A; \quad \partial_{X_A} F_{IB} = \partial_{X_B} F_{IA} \tag{5}$$

which usually arise in reducing higher order equations to a first order system.

By convenience and in order to get the right compatibility conditions of (3) and (4) we choose as new field variable (1)

$$\tilde{U} = (v, F_{IA}, \hat{\theta}, \theta_A)$$

requiring the mapping $V = V(U)$ to be locally invertible so that $\nabla_U V$ results a non singular matrix; that amounts to assume $\varepsilon_\hat{\theta} \neq 0$.

By requiring (4) to be consequence of (3), we easily obtain

$$\nabla_U f^A A_{nA} U = \tilde{U}' f - g \tag{6}$$

where $\nabla_U$ represents the gradient with respect to the field variables $U$, and $\tilde{U}'$ is the gradient of $\rho_0 h$ with respect to the components of $V$.

By the compatibility conditions (5), when are written explicitly in the field variable $U$ are exactly the same obtained in [2] by using a different procedure. If we consider isotropic materials, by making use of the results found in [2] on the representation theorem for isotropic functions it is possible to show that both the skew-symmetric tensors introduced in (9) are vanishing. The same is true for (Fourier type) materials [2]. But for these two special classes of materials the following relation is valid between the heat flux and the entropy flux:

$$h_A = U(0, 0) q_A \tag{10}$$

where $U(0, 0)$ is shown, [2], to be an «universal function» which at equilibrium reduces to $1/T = \Gamma_{|F}$ where $T$ is the absolute temperature.

Now, assuming that $h_A = q_A = 0$ when $\theta_A = 0$, we shall prove that a necessary and sufficient conditions in order the relation (10) to be valid for any anisotropic solid is that

$$\nabla U f^A A_{nA} = \tilde{U}' f - g \tag{11}$$

Because of the objectivity principle $h^A$ cannot depend on $v_i$, so we have:

$$\rho_0 \nabla_U h = \tilde{U}' = (\Gamma, \Gamma_{IA}, \Gamma, \eta_A, \mu);$$

given by

$$\Gamma_i = -(h_\hat{\theta}/\varepsilon_\hat{\theta}) u_i$$

$$\Gamma = h_\hat{\theta}/\varepsilon_\hat{\theta}$$

$$\Gamma_{IA} = \rho_0 h_{IA} - \rho_0 (h_\hat{\theta}/\varepsilon_\hat{\theta}) e_{FA}$$

$$\eta_A = \rho_0 (h_\hat{\theta}/\varepsilon_\hat{\theta}) e_{FA}$$

$$\mu = \rho_0 h_\hat{\theta} - \rho_0 (h_\hat{\theta}/\varepsilon_\hat{\theta}) e_{FA}$$

The relation (6) has to hold for every aritrary choice of $\partial_{X_A} U$ satisfying (5). Let us to stress now that the components of $U'$ play the same rule of the «Lagrange multipliers» in the theory of I-Shin Liu [6].

Carrying out explicitly the calculations one finds from (6) the following necessary and sufficient conditions of compatibility for (4) to be a consequence of (3):

$$\tilde{U}' \nabla_U f^A A_{nA} = \nabla_U h^A + K^A \tag{8}$$

where $K^A$, which results from the constraint (5) and from the symmetry conditions on $\theta_A$ and $F_{IA}$, is given by

$$K^A = (0, \Omega^A_{|B}, \Omega^A_{|B}, \kappa^A) \tag{9}$$

where $\Omega^A_{|B}, \omega^A_{|B}$ are completely arbitrary skew-symmetric tensor with respect to $A$ and $B$, while $\kappa^A$ is such that

$$\sum_A k^A A_{nA} = \tilde{U}' f - g.$$

The compatibility conditions (8), when are written explicitly in the field variable $U$ are exactly the same obtained in [2] by using a different procedure. If we consider isotropic materials, by making use of the results found in [2] on the representation theorem for isotropic functions it is possible to show that both the skew-symmetric tensors introduced in (9) are vanishing. The same is true for (Fourier type) materials [2]. But for these two special classes of materials the following relation is valid between the heat flux and the entropy flux:

$$\rho_0 \nabla_U h = \tilde{U}' = (\Gamma, \Gamma_{IA}, \Gamma, \eta_A, \mu);$$

which, when is written explicitly, gives us

$$d h^A = \Gamma v_i d(T_{IA}) - \Gamma_{IA} d u_i - \Gamma d(u_i T_{IA} - q_A) - \eta_A d \hat{\theta} +$$

$$- \Omega^A_{|B} d F_{IB} - \omega^A_{|B} d \theta_B - \kappa^A d \theta .$$

Because of the objectivity principle $h^A$ cannot depend on $u_i$, so we have:

\[ (i) \text{- denotes transposition.} \]