SOMMARIO: Si discute il problema dell'analisi elastoplastica dei mezzi isotopi dotati d'incrudimento lineare adottando le ipotesi della teoria della deformazione.

Vengono dimostrati due principi di estremo duali che consentono di ricordare la soluzione dei problemi al contorno alla minimizzazione di opportuni funzionali.

Il primo di tali principi, in assenza d'incrudimento, si riduce al ben noto teorema di Haar-Kármán che viene quindi generalizzato ai mezzi inerudenti.

Viene discussa anche l'estensione di quanto sopra accennato al caso d'incrudimento lineare a tratti e la Nota è chiusa da alcune considerazioni in merito alla connessione fra la teoria della deformatìone e la teoria incrementale dell'incrudimento cinematico.

SUMMARY: The paper discusses the boundary value problem for isotropic continua with elastic-linear hardening stress-strain laws of conventional deformation theory. A pair of dual extremum theorems are proven: the first, which takes as variables stresses and plastic strains, reduces the problem to the minimization of a quadratic functional subject to convex quadratic inequalities and linear equalities; the latter, which takes as variables displacements and plastic strains, reduces the problem to the minimization of a non-quadratic functional subject to linear equalities. In absence of hardening the first principle reduces to the well-known Haar-Kármán theorem for elastic perfectly plastic bodies. Some extensions of the above principles to piecewise linear hardening bodies and the connections of the deformation theory with the flow theory of linear kinematic hardening are reported at the end of the paper.

1. Introduction.

The deformation theory of plasticity is characterized among plasticity theories as the one in which relations between instantaneous states of stress and strain are postulated in such a way, that when the strain is given, the stress is uniquely determined or viceversa. This theory is generally regarded as physically unrealistic but practically useful for the study of boundary value problems in which the loading increases monotonically. Consequently, the deformation theory of plasticity is practically undistinguishable from a non-linear theory of elasticity, except when discontinuities in the slope of stress-strain relations are admitted. A well-known material which exhibits such a discontinuity is the Hencky material that can be considered in the deformation theory as the counterpart of the elastic-perfectly plastic material in the flow theories.

Several writers have derived extremal properties for the solution of the boundary value problem, assuming stress-strain relation of the deformation type without discontinuity in the slope.

The related extremum principles are essentially the well-known minimum principles of potential energy and complementary energy of elastic non-linear solids [1, 2, 3, 4, 5, 6, 7]. For the Hencky material, the Haar-Kármán principle [8] reduces the search for the stresses in the boundary value problem to a convex optimization problem.

Working with yield surfaces consisting of independent hyperplanes in stress space which harden kinematically, under the assumption of regularly progressive yielding, Maier [9, 10, 11] has shown that boundary value problem is reducible to a quadratic, linear-inequality-constrained optimization problem.

The advantages of such a principle might be attributed mainly to the circumstance that minimization of quadratic forms, subject to linear inequalities, is a fairly tractable problem, which reduces, after some discretization, to a quadratic programming problem for whose solution several efficient numerical algorithms are available. Compared with the classical extremum principles mentioned above, Maier's theorems deal with more numerous unknowns connected with the number of hyperplanes that constitute the yield polyhedron, but present the advantages mentioned above. On the other hand, the recent developments in the theory and technique of convex-inequality-constrained optimization suggest that some quite simple principles, such as the Haar-Kármán theorem, can not only be of theoretical interest but also of practical usefulness.

The principle referred to does not, however, consider the influence of hardening, and it is therefore restricted to elastic-perfectly plastic isotropic material.

In this paper we are considering the boundary value in isotropic continua with elastic-linear hardening stress-strain relations of the classical deformation theory.

It will be shown that this problem is equivalent to the minimization of a quadratic functional subjected to a convex quadratic inequality. This procedure, in the absence of hardening, reduces to the classical Haar-Kármán theorem. Then a dual extremum principle which reduces the same problem to the minimization of inequality-unconstrained non-quadratic functional is also proven. Some
2. Basic assumptions.

The finite relations between stress and strain in the conventional deformation theory are based on the following assumptions:

a) The body is isotropic

b) The relative volumetric change \( \varepsilon = (1/3) \delta_{ij} \varepsilon_{ij} \) is an elastic deformation proportional to the mean pressure \( \sigma = (1/3) \delta_{ij} \sigma_{ij} \) viz:

\[
e = \frac{\sigma}{3K}
\]

where \( K \) is the bulk modulus:

\[
K = \frac{E}{3(1-2\nu)}.
\]

c) The strain deviatoric components:

\[
\varepsilon_{ij} = \varepsilon_{ij} - \delta_{ij} \varepsilon
\]

and the stress deviatoric components:

\[
\sigma_{ij} = \sigma_{ij} - \delta_{ij} \sigma
\]

are proportional, viz:

\[
\varepsilon_{ij} = \mu \sigma_{ij}.
\]

Hence it follows immediately that if we put:

\[
S = \sqrt{\varepsilon_{ij} \varepsilon_{ij}},
\]

\[
\Gamma = \sqrt{\sigma_{ij} \sigma_{ij}}
\]

from Eq. (5) we can write:

\[
\Gamma = \mu S.
\]

Eliminating the scalar \( \mu \) from Eqs. (5) and (7), we find:

\[
\varepsilon_{ij} = \frac{\Gamma}{S} \cdot \sigma_{ij}
\]

The above equations are incomplete since they contain the unknown function \( \Gamma \); to determine the latter, a supplementary relation of the following type:

\[
\Gamma = \Gamma(S)
\]

is required. To this end, we introduce the further assumption:

d) The stress-strain deviation curve \( S - \Gamma \) is assumed to be of the elastic-plastic type with linear hardening, as shown in Fig. 1.

The bilinear law shown in Fig. 1 can be defined by the quantities:

\[
S_0, \quad \tan \alpha = 2G, \quad \tan \beta = 2G_t < 2G
\]

where \( G \) is the shear modulus of the linear elastic behaviour and \( G_t \) the tangent shear modulus of the assumed linear hardening behaviour.

Strain deviation \( \Gamma \) corresponding to a stress deviation value \( S \), can be written as the sum of the elastic strain deviation:

\[
\Gamma^e = S \cot \alpha
\]

and the corrective strain deviation:

\[
\Gamma^c = (S - S_0)(\cot \beta - \cot \alpha)
\]

where the "hardening modulus" \( C \) is expressed by:

\[
C = \frac{2G G_t}{G - G_t}.
\]

In accordance with Eq. (8), the strain deviatoric components \( \varepsilon_{ij} \) can be written as the sum of the elastic strains:

\[
\varepsilon_{ij} = \frac{\Gamma^e}{S} \cdot \sigma_{ij} = \frac{1}{2G} \cdot \sigma_{ij}
\]

and the corrective strains:

\[
\varepsilon_{ij}^c = \frac{\Gamma^c}{S} \cdot \sigma_{ij} = \frac{S - S_0}{C \sigma} \cdot \sigma_{ij}
\]

if \( S \geq S_0 \)

\[
\varepsilon_{ij}^c = 0
\]

if \( S < S_0 \)

If we put:

\[
\sigma_{ij}' = \sigma_{ij} - C \sigma_{ij}
\]

from Eqs. (14) and (17) we easily obtain:

\[
\sigma_{ij}' = \frac{S_0}{S} \sigma_{ij}
\]

if \( S \geq S_0 \)

\[
\sigma_{ij}' = \sigma_{ij}
\]

if \( S < S_0 \)

Fig. 1.