SUMMARY: In this paper two cases of transverse disturbances in plane Poiseuille flow with quite different Reynolds number R have been considered. The aim is to prove the feasibility of a computational method described by the author in a previous paper. This method can be applied to the study of non-linear two- and three-dimensional disturbances in plane parallel flows.

One of the cases chosen as a check is the well known Thomas case with $R = 10^4$ and $\lambda = 1$. The other case is $R = 250$ and $\lambda = 0.5$ and its interest lies in the fact that the Reynolds stress has a sign opposite to viscosity.

After having written down in Section 1, some formulae useful for comparison, the Thomas case is considered in Section 2. Some space is given to discussing Thomas values in order to show the importance of using three-point instead of five-point finite-difference expressions for second derivatives across the channel.

Reynolds stress, stream function, vorticity, rate of amplification and celerity of the perturbation, and separate contributions to the last two quantities from viscosity and transport result from the method in full details across the channel for this and for the following case in Section 3.

The feasibility and accuracy of the method may thus be regarded as proven.

1. Introduction and general formulae.

The numerical approach used here refers to disturbances in plane parallel flows and is somehow like a simulation. Perturbation waves, which are assumed to be periodic in the mean motion direction, are allowed to settle with time both in phase and amplitude anywhere across the flow. This means solving a system of non-linear partial differential equations with respect to time and to the space-coordinate orthogonal to the wall(s). The general method for non-linear transverse and longitudinal disturbances has been described in a previous paper [1] in the following referred as $A$.

In the present paper the method has been checked for Poiseuille flow in the well known unstable linear case studied by Thomas [2] having Reynolds number $R = 10^4$ and wave number for a $2\pi$ length $\lambda = 1$.

The other case is a very stable one corresponding to $R = 250$ and $\lambda = 0.5$, and some results for this last case were already given in $A$ with a description based on very few points across the channel. Repeating now the computation with a higher number of points it turns out that previous results in $A$ were significant. In particular, the conclusion that the effect of the Reynolds stress is in this case opposite to viscosity is confirmed here.

These two linear cases are thought sufficient here for checking purposes, as the effect of the amplitude can simply be allowed for in the equations and does not affect the computational method.

As to the accuracy, special attention has been given to discussing the use of a three-point expression for second derivatives across the channel. It is concluded that the three-point one is as good as the five-point one, in the sense that where the first is not sufficient the same is true for the other one and the only help would be to increase the number of points across the channel.

Numerical results agree with this conclusion as the disturbance is found unstable and the increase of amplitude per unit time is found to coincide with Thomas’s. The same is true for the celerity of the wave, i.e. for the length travelled by the perturbation in the unit time.

As to the formulae, we shall now rewrite first from $A$ some fundamental ones and also express them in a form useful for comparison with numerical results.
The motion has been characterised assuming that the velocity components can be written as

\[ u(x, y, z, t) = u_0(x, t) + u_1(x, t, y) + u_2(x, t, z) + u_3(x, t, x, z) \]

\[ v(x, y, z, t) = v_1(x, t, y) + v_2(x, t, z) + v_3(x, t, x, y) \]

\[ w(x, y, z, t) = w_1(x, t, y) + w_2(y, t, z) + w_3(x, t, x, y) \]

(1)

where \( u_0(x, t) \) describes the basic flow, which is in the \( x \) direction, and other contributions are intended to belong to disturbances. Functions with index 1, 2 or 3 are at any instant \( t \) periodic with respect to \( x \) or \( z \) or both these variables, and are of zero mean value.

In the case of only transverse disturbances one can consider only \( u_1(x, t, y) \) and \( v_1(x, t, y) \) as different from zero, and it is worth using a stream function, writing

\[ n_1(y, t, x) = \frac{\partial}{\partial y} \psi(y, t, x) \]

\[ v_1(y, t, x) = -\frac{\partial}{\partial x} \psi(y, t, x) \]

(2)

Navier-Stokes equations are reduced for this disturbance to a single vorticity equation, namely

\[ \frac{\partial \zeta}{\partial t} = n_0 \frac{\partial^2 \zeta}{\partial x^2} - n_1 \frac{\partial^2 \zeta}{\partial x \partial y} + \frac{1}{R} \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \zeta - \left[ n_1 \frac{\partial \zeta}{\partial x} - n_1 \frac{\partial \zeta}{\partial y} + v_1 \frac{\partial \zeta}{\partial y} - v_1 \frac{\partial \zeta}{\partial x} \right] \]

(3)

where bars represent mean values with respect to the variables indicated aside, and

\[ \zeta = - \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \psi(y, t, x) \]  

(4)

As to the units employed, equations have been made nondimensional expressing all quantities in terms of an arbitrary length \( D \) and a velocity \( U \). The ratio \( D/U \) has been used as unit of time. Reynolds number is \( R = DU/\nu \), \( \nu \) being the kinematic viscosity.

For the main flow we have:

\[ \frac{\partial u_0(y, t)}{\partial t} = -q_0(y, t) + \frac{1}{R} \frac{\partial^2}{\partial y^2} u_0(y, t) - \frac{\partial}{\partial y} (v_1 u_1) \]

(5)

\( q_0(y, t) \) arising from the mean pressure.

Assuming furthermore the perturbation to be sinusoidal, i.e.

\[ \psi(y, t, x) = \Phi(y, t) \cos(\lambda x) + \Phi_2(y, t) \sin(\lambda x) \]

\[ \zeta_1(y, t, x) = \chi(y, t) \cos(\lambda x) + \Phi_2(y, t) \sin(\lambda x) \]

(6)

the last term of Eq. (3) can be neglected and the equations of motion written:

\[ \frac{\partial n_0(y, t)}{\partial t} = \frac{1}{R} \frac{\partial^2 n_0(y, t)}{\partial y^2} - q_0(y, t) - \frac{1}{2} \left( \frac{\partial^2 \chi}{\partial y^2} - \frac{\partial^2 \Phi}{\partial y^2} \right) \]

\[ \frac{\partial \chi}{\partial t} = \frac{1}{R} \left( \frac{\partial^2}{\partial y^2} \chi - \lambda^2 \chi \right) - \lambda \left( u_0 \chi + \frac{\partial^2 u_0}{\partial y^2} \right) \]

\[ \frac{\partial \Phi}{\partial t} = \frac{1}{R} \left( \frac{\partial^2}{\partial y^2} \Phi - \lambda^2 \Phi \right) + \lambda u_0 \Phi + \frac{\partial^2 u_0}{\partial y^2} \Phi \cos(\sigma - \tau) \]

(7)

The pure number \( \lambda \) represents the number of waves of the disturbance in a length \( 2\pi D \).

In actual computations Eqs. (7)1,2,3,4,5 have been divided by \( \lambda \) and \( \tau / \lambda \) has been used as computing time. Its unit \( D/\lambda U \) is the time required for travelling with velocity \( U \) along a length equal to \( D/\lambda \) i.e. equal to the wave length divided by \( 2\pi \).

Boundary conditions in Poiseuille flow are at each wall

\[ \psi_1 = \psi_2 = 0 \quad \frac{\partial \psi_1}{\partial y} = \frac{\partial \psi_2}{\partial y} = 0 \]

(8)

meaning that there both components of velocity are zero for the disturbance.

In Eqs. (7) we can distinguish Eqs. (7)2 and (7)3 as true motion equations for the disturbance and Eqs. (7)4 and (7)5 as definition relations between its vorticity and its stream function.

Motion equations (7)2 and (7)3 can be transformed as

\[ \frac{\partial \chi}{\partial t} = \frac{1}{R} \left[ \frac{\partial^2 \chi}{\partial y^2} - \frac{\partial^2}{\partial y^2} \right] - \lambda \left( \frac{\partial^2 u_0}{\partial y^2} \Phi \sin(\sigma - \tau) \right) \]

(9)

obtained by writing instead of Eqs. (6) the following:

\[ \psi(y, t, x) = \Phi(y, t) \cos(\lambda x - \sigma(y, t)) + \psi_2(y, t) \sin(\lambda x) \]

\[ \zeta_1(y, t, x) = \chi(y, t) \cos(\lambda x - \tau(y, t)) + \Phi_2(y, t) \sin(\lambda x) \]