THE STABILITY PROBLEM FOR COUETTE FLOW: 
A FINITE DIFFERENCE APPROACH 
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SOMMARIO: In questa nota vengono studiati i vortici di Taylor facendo uso dei metodi di approssimazione alle differenze finite; si danno dettagli sullo schema di soluzione numerica, accennando alle difficoltà di convergenza talvolta incontrate ed ai procedimenti con i quali tali difficoltà sono state superate. I risultati ottenuti (in particolare per il caso di cilindri controrotanti) sono riuniti in un certo numero di figure.

SUMMARY: Taylor vortices are studied using finite difference approximations. Details of the numerical scheme of solution are given. Mention is made of convergence difficulties which are sometimes encountered and of the procedure with which the difficulties can be overcome. Results, particularly in the case of counter-rotating cylinders, are shown in a number of graphs.

1. Introduction.

We report here results of work on a numerical study of the problem of instability of Couette flow as a consequence of the onset of Taylor vortices precisely we consider the steady state reached after transition (for earlier work see [1], [2] and references there). In our approach we use finite-difference methods to obtain approximations to the solutions of the non-linear equations which describe steady-state perturbations of the Couette flow. Other techniques could be also successful in dealing with the special problem (see [3], [4] and references there), but we were interested in experimenting with finite differences, because of their adaptability to a great variety of circumstances. The main question in our approach concerns the convergence of the procedures of successive approximation adopted to deal with non-linear terms. For convergence a certain limit on the meshsize must not be trespassed but there are other difficulties (convergence may occur towards the solution of a problem, different from the one originally stated and devoid of physical meaning); on these difficulties and on numerical devices that can be used to overcome them we report in the paper.

2. Position of the problem.

We consider, the steady flow of a viscous incompressible fluid kinematic (viscosity \( \nu \)) in the unbounded clearance between two coaxial rotating cylinders of radii \( R_1, R_2 \) (\( R_1 < R_2 \)); the angular speeds of the two cylinders are respectively \( \Omega_1 \) and \( \Omega_2 \). As is well known, under some circumstances, the observed flow is not the fundamental Couette flow but rather a system of steady vortices. Hence the interest of the study of perturbations to the Couette flow, which are steady, periodic in the axial direction and axially symmetric.

We approach the problem in a formulation which is stated precisely below, by which can be roughly described as follows: starting from the Navier-Stokes equations in cylindrical coordinates, we obtain a differential system involving: (i) a function \( \nu \) proportional to the perturbation in the transverse component of speed (perturbation which is counted from the value relating to Couette flow); (ii) a stream function \( \psi \) for radial and axial components of speed.

The differential problem is as follows:

\[
\Delta \psi = \frac{2\varepsilon}{\varepsilon + 1} \left( \frac{\partial^2 \psi}{\partial \xi^2} + \frac{\partial^2 \psi}{\partial \eta^2} \right) + \frac{3\varepsilon^2}{(\varepsilon + 1)^2} \frac{\partial \psi}{\partial \xi} + \frac{3\varepsilon^3}{(\varepsilon + 1)^3} \frac{\partial \psi}{\partial \xi} = 2T \frac{\partial \psi}{\partial \xi} \left( \frac{\xi + 1}{\varepsilon + 2} \right) + \frac{1 - \lambda}{\varepsilon + 1} + (\varepsilon + 1) \left( \frac{1}{\varepsilon + 1} \right) \psi + N_t(\psi);
\]

\[
\psi(0, \xi) = \psi(1, \xi) = 0;
\]

\[
\psi(\xi, -q) = \psi(\xi, q) = 0;
\]

\[
V(0, \xi) = V(1, \xi) = 0;
\]

\[
\frac{\partial V}{\partial \xi}(\xi, -q) = \frac{\partial V}{\partial \xi}(\xi, q);
\]

Here the parameters have the following significance

\[
\varepsilon = \frac{R_2 - R_1}{R_1}, \quad T = \frac{R_1 \Omega_2^2}{\nu^2}, \quad \lambda = \frac{\Omega_2}{\Omega_1};
\]

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and 2q(R_2 - R_1) is the axial period of the perturbation in dimensional coordinates.

The physical components of the speed in the fluid $u, v, w$ are related to $\psi$ and $V$ through the formulae

$$ u(r, \zeta) = -\frac{v}{R_3 e(e\xi + 1)} \frac{\partial \psi}{\partial \zeta}; $$

$$ w(r, \zeta) = \frac{v}{R_3 e(e\xi + 1)} \frac{\partial \psi}{\partial \xi}; $$

$$ v(r, \zeta) = \Omega_1 R_1 \left( \frac{\epsilon + 1}{\epsilon^2 + 2} \frac{1}{\epsilon} \left( 1 - \frac{\lambda}{\epsilon e} \right) + V(\xi, \zeta) \right); $$

whereas the variables $\xi$ and $\zeta$ are related to cylindrical coordinates $r, \theta$ as follows

$$ \xi = \frac{r - R_1}{R_2 - R_1}; \quad \zeta = \frac{\zeta}{R_2 - R_1}; $$

$N_1(\psi, \psi, V)$ and $N_2(\psi, V, V)$ are quadratic terms; they do not depend on $\lambda$ and $T$, and are such that for $\epsilon = 0$ they reduce respectively to the Jacobians

$$ \frac{\partial (\psi, \psi)}{\partial (\xi, \zeta)}; \quad \frac{\partial (\psi, V)}{\partial (\xi, \zeta)}; $$

$\psi = 0, V = 0$ is the solution of (2.1) which corresponds to undisturbed Couette flow; the existence of non-trivial solutions indicates the physical possibility of a flow different from the Couette flow.

Our task is that of finding approximate non-trivial solutions of (2.1). There are many examples of such solutions; as we have mentioned, our interest is in discrete methods.

Before we pass on to the finite-difference version of (2.1) it is convenient to change the axial variable $\zeta$ into

$$ Z = \frac{\zeta}{q}, $$

so that our problem is now posed in the domain

$$ Q: 0 \leq \xi \leq 1, \quad -1 \leq Z \leq 1, $$

which is independent of $q$.

Finite differences are based on a square mesh on $Q$ of size $b = 1/(n + 1)$ ($n, a$ positive integer) and on the use of central differences for differentials. Taking also the boundary conditions into account, the equations become

$$ U_1 \Psi = b^2 q TB_1 V + TM_{1,1}(V) + M_{1,3}(\psi), \quad (2.2) $$

$$ U_2 V = b q B_2 \psi + M_2(\psi, V), $$

where:

- a) $\Psi, V$ are two $(2n^2 + 2n)$-vectors, whose components are the values taken by two nodal functions $\psi, \psi$ on the internal mesh-points of $Q$, ordered from left to right and from top to bottom;

b) $U_1, U_2, B_1, B_2$ are the following block-circulant matrices of order $2n + 2$ in blocks of order $n$:

$$ U_1 = \{ A, B, I, O, ..., O, I, B \}; $$

$$ U_2 = \{ C, I, O, ..., O, I \}; $$

$$ B_1 = \{ O, -D_1, O, ..., O, D_1 \}; $$

$$ B_2 = \{ O, -D_2, O, ..., O, D_2 \}; $$

where $A, B, C$ are $n \times n$-matrices whose elements are all zero except:

$$ a_{1,1} = 7q^4 + 8q^3 + 6 + \frac{eb q^3}{\epsilon b + 1} \left( 1 - \frac{eb}{\epsilon b + 1} \right); $$

$$ a_{i,1} = 6q^4 + 8q^3 + 6 - \frac{eb q^3}{\epsilon b + 1} \left( 1 + \frac{eb}{\epsilon b + 1} \right); $$

$$ a_{i,n} = 7q^4 + 8q^3 + 6 - \frac{eb q^3}{\epsilon b + 1} \left( 1 - \frac{eb}{\epsilon b + 1} \right); $$

$$ a_{i,i+1} = (2q^2 + 2q^2) \left( 2 + \frac{eb}{\epsilon b + 1} \right) + 3q^4 \left( \frac{eb}{\epsilon b + 1} \right)^2 \cdot \left( 1 - \frac{1}{2} \frac{eb}{\epsilon b + 1} \right); $$

$$ a_{i,i+2} = q^4 \left( 1 - \frac{eb}{\epsilon b + 1} \right); \quad i = 1, 2, ..., n - 2; $$

$$ a_{i,i-2} = q^4 \left( 1 + \frac{eb}{\epsilon b + 1} \right); \quad i = 3, 4, ..., \#; $$

$$ b_{i,j} = -4(q^2 + 1); \quad i = 1, 2, ..., \#; $$

$$ b_{i,i+1} = q^2 \left( 2 - \frac{eb}{\epsilon b + 1} \right); \quad i = 1, 2, ..., \#; $$

$$ b_{i,i-1} = q^2 \left( 2 + \frac{eb}{\epsilon b + 1} \right); \quad i = 2, 3, ..., \#; $$

$$ c_{i,i} = -2(q^2 + 1) - q^2 \left( \frac{eb}{\epsilon b + 1} \right)^2; \quad i = 1, 2, ..., \#; $$

$$ c_{i,i+1} = q^2 \left( 1 + \frac{eb}{2(\epsilon b + 1)} \right); \quad i = 1, 2, ..., \#; $$

$$ c_{i,i+1} = q^2 \left( 1 + \frac{eb}{2(\epsilon b + 1)} \right); \quad i = 1, 2, ..., \#; $$

$D_1$ and $D_2$ are diagonal matrices of order $n$ with the following principal elements

$$ d_{i,j} = \frac{1}{\epsilon b + 1} \frac{ieb + 2}{ieb + 1} \frac{1 - i(1 + \epsilon)^2}{\epsilon + 2}; $$

$$ d_{i,j} = \frac{1}{ieb + 1}; $$

$I$ is the identity matrix of order $n$.

c) $M_{1,1}, M_{1,2}, M_2$ are non-linear operators; $M_{1,1}(M_{1,2})$