A GENERAL METHOD FOR THE INCREMENTAL SOLUTION OF ELASTIC-PLASTIC PROBLEMS*

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SUMMARY: A general iterative method for the incremental solution of elastic-plastic problems is described. The convergence of the procedure on the rigorous solution is demonstrated.

1. Introduction.

It was shown in a previous note [1] that a field minimum principle involving a quadratic functional, which generalises in differential terms for elastic-plastic continua the well-known principle of the minimum total potential energy of elastic bodies, could be used for the incremental solution of elastic-plastic problems.

The present note aims to show how in the light of that principle we can arrive at the incremental solution of elastic-plastic problems via an elementary iterative method the sure convergence of which is proven.

The minimum principle referred to concerns the determination of the variations in displacement $u_i$ and in plastic dislocation $t_i$ that occur in any given elastic-plastic continuum when it is subjected to variations in body forces $X_i$, in volume $V$, in surface tractions $p_i$ on portion $S_p$ of the boundary and in surface displacements $u_{i0}$ on portion $S_u$ of the boundary, starting from a situation in which $X_i$, $p_i$, $u_{i0}$ represent the present values of these actions.

It is assumed that the stress state $a_{ij}$ corresponding to the above values $X_i$, $p_i$, $u_{i0}$ of the external stresses is known and that therefore the division of volume $V$ of the body into the elastic part $V_e$ and the part at the threshold of plastic flow $V_p$ is known a priori. Indeed, assuming that a yield condition, fixed or evolving, according to whether the material is perfectly plastic or subject to work-hardening, has been established by the relation:

$$f(a_{ij}) = 0 \quad (1)$$

knowledge of the stress state $a_{ij}$ enables us to identify the aliquots $V_e$ and $V_p$ of volume $V$ by means of the following relations:

$$f(a_{ij}) < 0 \quad \text{in } V_e$$

$$f(a_{ij}) = 0 \quad \text{in } V_p$$

and further to associate to any set:

$$\dot{u}_i = \dot{u}_i(x_i) \quad \text{defined in } V$$

$$\dot{t}_i = \dot{t}_i(x_i) \quad \text{defined in } V_p$$

a single well-defined distribution of elastic strain increments $\varepsilon_{ij}$ and stress increments $\sigma_{ij}$.

I would recall that, assuming small displacements in the known sense, we have:

$$\varepsilon_{ij} = \frac{1}{2} \left( \frac{\partial \dot{u}_i}{\partial x_j} + \frac{\partial \dot{u}_j}{\partial x_i} - \frac{\partial f}{\partial 

\sigma_{ij}} \right) \frac{\partial \dot{t}_i}{\partial \sigma_{ij}} \lambda \quad (4)$$

in the case of perfectly plastic material, and:

$$\varepsilon_{ij} = \frac{1}{2} \left( \frac{\partial \dot{u}_i}{\partial x_j} + \frac{\partial \dot{u}_j}{\partial x_i} - b \frac{\partial f}{\partial \sigma_{ij}} \right) \lambda \quad (5)$$

where $b$ is a known positive definite function of $\sigma_{ij}$, in the case of work-hardening.

Equations (4) or (5) thus define $\varepsilon_{ij}$ unequivocally for every set of variables $\dot{u}_i$, $\dot{t}_i$.

The variation of the stress state $\sigma_{ij}$ is then unequivocally determined in the form:

$$\sigma_{ij} = B_{ijkl} \varepsilon_{kl} \quad (6)$$

as soon as the tensor of elastic constants $B_{ijkl}$ is known.

The minimum principles to which I am referring assert that of all the possible sets of variables (3) chosen that comply with the conditions:

$$\dot{u}_i = \dot{u}_{i0} \quad \text{in } S_u$$

$$\dot{t}_i = 0 \quad \text{in } V_p$$

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(1) It is assumed that the yield surface is of the regular type. This hypothesis is not, however, restrictive for the reasons given later and will in fact be removed in para. 4.
the one corresponding to the effective solution of the incremental problem minimises the expression:
\[ M_p = \frac{1}{2} \int_V \dot{\varepsilon}_{ij} \dot{e}_{ij} dV + \int_{S_p} \dot{\varepsilon}_{ij} \dot{b}_{ij} dS \]  
(8)

if the material is perfectly plastic, or the expression
\[ M_t = \frac{1}{2} \int_V \dot{\varepsilon}_{ij} \dot{e}_{ij} dV + \frac{1}{2} \int_V \dot{\varepsilon}_{ij} \dot{b}_{ij} dV - \int_{S_p} \dot{\varepsilon}_{ij} \dot{b}_{ij} dS \]  
(9)

if the material is subject to work-hardening.

The difficulty in the concrete application of such principles lies in the fact that, as we have to look for the minimum with the sign constraint \( \dot{\lambda} \geq 0 \), we have to determine a field minimum and not an analytic minimum.

It was in fact seen in [1] that only on the assumption that elastic unloading could be regarded as negligible could the condition:
\[ \dot{\lambda} \geq 0 \quad \text{in} \quad V \]  
(10)

be substituted for the second of conditions (7).

In that case we have to seek the analytic minimum of expressions (8) or (9) and the problem is reduced to solving a linear differential system which there is no point in detailing here.

The purpose of this note is, as already mentioned, to supply an easily applicable iterative method of sure convergence for determining the field minimum of the functionals (8) and (9), that is the rigorous solution of the elastic-plastic incremental problem.

2. The case of an elastic-perfectly plastic body.

The field minimum of functional (8) corresponds to the determination of set \( u_i, \lambda \) that satisfies the following relations:

\[ \frac{\partial \sigma_{ij}}{\partial x_j} + \dot{X}_i = 0 \quad \text{in} \quad V \]  
(11a)

\[ \dot{\sigma}_{ij}\eta_{ij} = \dot{p}_i \quad \text{in} \quad S_p \]  
(11b)

\[ \dot{\mu}_i = \dot{u}_0 \quad \text{in} \quad S_u \]  
(11c)

\[ \dot{\lambda} = 0 \quad \text{in} \quad V_e \]  
(11d)

\[ \begin{cases} \dot{\lambda} = 0 & \text{if} \quad \dot{f} < 0 \\ \dot{\lambda} = 0 & \text{if} \quad \dot{f} = 0 \end{cases} \quad \text{in} \quad V_p. \]  
(11e)

The solution, that is, must be equilibrated both inside (11a) and on the boundary (11b) of the solid; it must be compatible with the external constraints (11c) and, lastly, must conform to the plastic constitutive laws (11d) and (11e).

To translate relations (11) into terms of displacement rate \( \dot{u}_i \) and plastic dislocation rate \( \dot{\lambda} \), note that in virtue of (4) and (6) the following holds:
\[ \dot{\sigma}_{ij} = B_{ijke} \left( \frac{\partial \dot{u}_k}{\partial x_e} + \frac{\partial \dot{u}_e}{\partial x_k} \right) - B_{ijke} \frac{\partial f}{\partial \sigma_{ke}} \dot{\lambda}. \]  
(12)

Bearing in mind the properties of symmetry:
\[ B_{ijke} = B_{jike} = B_{kije} = B_{keij} \]  
(13)

of the tensor of the elastic constants, (12) may also be written in the form:
\[ \dot{\sigma}_{ij} = B_{ijke} \frac{\partial \dot{u}_k}{\partial x_e} - B_{ijke} \frac{\partial f}{\partial \sigma_{ke}} \dot{\lambda}. \]  
(14)

which, taking the position
\[ B_{ij} = B_{ijke} \frac{\partial f}{\partial \sigma_{ke}} \]  
(15)

assumes the form:
\[ \dot{\sigma}_{ij} = B_{ijke} \frac{\partial \dot{u}_k}{\partial x_e} - B_{ij} \dot{\lambda}. \]  
(16)

It is worth noting, as will be deduced from (15), that as the pre-existing stress state \( \sigma_{ij} \) is known, tensor \( B_{ij} \) is likewise known.

It is necessary at this point to develop the expression for the variation \( \dot{f} \) of the yield condition \( f \). To that end, having noted that:
\[ \dot{f} = \frac{\partial f}{\partial \sigma_{ke}} \dot{u}_k \]  
(17)

it follows from (16) that:
\[ \dot{f} = \frac{\partial f}{\partial \sigma_{ke}} \left( B_{keij} \frac{\partial \dot{u}_i}{\partial x_j} - B_{ke} \dot{\lambda} \right) \]  
(18)

and so we have in conformity with (15):
\[ \dot{f} = B_{ij} \frac{\partial \dot{u}_i}{\partial x_j} - B_{ke} \frac{\partial f}{\partial \sigma_{ke}} \dot{\lambda}. \]  
(19)

Thus if we have the position:
\[ B = B_{ke} \frac{\partial f}{\partial \sigma_{ke}} \]  
(20)