LARGE AMPLITUDE INTERNAL GRAVITY WAVES IN STABLY STRATIFIED FLUIDS: LINEAR MODEL

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1. Introduction.

The general study of fluid motions is highly complex. No existing theories explain the phenomenon of inhomogeneous turbulence. And yet this is a feature of most fluid mechanical problems. The demarcation between laminar and turbulent motions is a stability problem, for which simplified theories exist. When a fluid is stably stratified in density or velocity, a disturbance in this fluid creates complex internal motions. A special feature is the phenomenon of large amplitude internal waves, interspersed with regions of rotors or turbulence. This is the phenomenon under study here.

Research in this field has been inspired by meteorologists, since the atmosphere is a stratified fluid. (See Scorer, 1949).

In this paper, we consider a linearised model of a stratified, incompressible fluid suitable for laboratory scale experiments. The equations are generalised by Fourier analysis in order to consider motions past an obstacle on the surface. This may be regarded as an analogue of the general atmospheric problem.

2. Theory.

The motion equation used is a simplified form of the Navier-Stokes equation:

\[(v \cdot g \text{ rad})v = -\frac{1}{\rho} g \text{ rad } p + g .\]

We have assumed a steady state in an inviscid fluid. It should be stressed that this could lead to considerable error; but this simplified model could, nevertheless, be useful to explain some of the phenomena under observation here.

The vorticity equation is obtained directly:

\[
curl [\rho \{ (v \cdot g \text{ rad})v - g \}] = 0 .
\]

In two-dimensional flows, only the y-component survives, and we have:

\[(v \cdot g \text{ rad}) \omega_y = [g \text{ rad } (\log \rho) \times (g - (v \cdot g \text{ rad})v)]_y .\]

With the help of some vector algebra, we may arrive at:

\[
\frac{\partial \omega_y}{\partial y} = \frac{\partial}{\partial y} \left[ \left( \frac{\partial}{\partial x} \ln \rho \right) \times \left( g - (v \cdot g \text{ rad})v \right) \right].
\]

Defining

\[\beta_0 = -\frac{1}{\rho} \frac{\partial \rho}{\partial z},\]

and is known as the static stability.

Defining

\[G_0 = \frac{g \beta_0}{U_0^2},\]

\[R_0 = \frac{1}{2} U_0,\]

and noting that far upstream the streamlines are horizontal and undisturbed, we write the wave equation in the form:

\[\nabla^2 \xi_0 + \frac{R_0}{R_0} (\nabla \xi_0)^2 - G_0 \xi = \frac{R_0}{R_0} - G_0 \xi_0 .\]

Defining \(\delta = z - \xi_0\), the equation for \(\delta\) is

\[\nabla^2 \delta + \frac{1}{2} \left( \nabla \delta \right)^2 - 2 \frac{\delta \delta}{\partial \xi} \frac{d}{d \xi} (\ln U_0^2) + \delta G_0 \delta = 0 .\]
This equation can be rendered linear if we choose the conditions:

\[ U_0^2 = \text{constant} \]

and

\[ G_0 = \text{constant} . \]

We then arrive at the linear equation:

\[ \nabla^2 \delta + G_0 \delta = 0 . \quad (1) \]

3. Solution by Fourier transforms.

Most theoretical literature written on this subject has in fact depended on the linearised equation.

Writing the transform \( \tilde{\delta} \) for \( \delta = \delta(x, z) \),

\[ \tilde{\delta} = \int_0^\infty e^{-ikx} \delta \ dx \quad (2) \]

we transform Eq. (1) to the following:

\[ -k^2 \tilde{\delta} + \frac{d^2 \tilde{\delta}}{dz^2} + G_0 \tilde{\delta} = 0 . \]

This has the solution

\[ \tilde{\delta} = A \sin rvz + B \cos rvz \quad (3) \]

where

\[ r = \sqrt{G_0 - k^2} \quad (4) \]

(i) At \( z = 0 \), \( \tilde{\delta} = 0 \), \( B = 0 \)

\[ \therefore \tilde{\delta} = A \sin rvz \quad (5) \]

(ii) At \( z = H \),

\[ \tilde{\delta}_{H,k} = A \sin rH \]

\[ = \int_0^\infty e^{-ikx} \delta_{(x,H),z} \ dx \]

\[ = \tilde{F}(k) \ (\text{say}) \quad (6) \]

We therefore have:

\[ \tilde{\delta}_{z,k} = \frac{\tilde{F}(k) \sin rvz}{\sin rvH} . \quad (7) \]

Hence

\[ \tilde{\delta}_{z,k} = \int_0^\infty \frac{\tilde{F}(k) \sin rvz}{\sin rvH} e^{ikz} \ dk . \quad (8) \]

The solution of this integral depends on a suitable choice of \( F(k) \). Queney, Scorer and others have used a ridge profile of shape:

\[ \delta_{H,k} = -a^2 \int_0^\infty e^{ikx-kb} \ dx \]

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\[ = -a^2 \int_0^\infty \frac{\sin rvz}{\sin rvH} e^{ikz} \ dk . \quad (9) \]

This reduces the problem to the solution of the following integral:

\[ \delta_{z,k} = -a^2 \int_0^\infty \frac{\sin rvz}{\sin rvH} e^{ikz} \ dk \]

(10)

If \( \sin rH = 0 \), the integral is \( \infty \), i.e. \( rH = n\pi \). Hence the poles of this integral correspond to the solution when there is no corrugation on the boundary. Corresponding to each wave train on the boundary, there will be internal waves, and if the shape of the ridge is composed of components of all the wavelengths, then the wavelength which corresponds to zero amplitude on the boundary, will produce "resonant" lee waves.

After Scorer, the evaluation of (10) has been effected approximately by contour integration.

From Cauchy's theorem, the integral round a closed contour \( C \), \[ \int_C f(k) dk \] is given by \( 2\pi i \) times the sum of the residues at the poles inside \( C \).

The residue at the pole may be calculated as follows: If

\[ f(k) = \frac{g_1(k)}{g_2(k)} \]

the pole exists where \( g_2(k) = 0 \). If \( k = k^* \) at the pole, then the residue is given by

\[ R = \frac{g_1(k^*)}{g_2(k^*)} . \]

It is assumed that the pole is of the first order, and that the formula is not general. A more rigorous mathematical argument would not necessarily serve a better purpose, as the theory of this problem is at best an approximation to the physical features encountered in practice.

From Cauchy's theorem, we have:

\[ \tilde{\delta}_{(x,k)} = -a^2 \int_0^\infty \frac{\sin rvz}{\sin rvH} e^{ikz} \ dk \]

\[ = -a^2 \int_0^\infty \frac{\sin rvz}{\sin rvH} e^{ikz} \ dk \]

\[ + \frac{\pi i a^2 e^{-k^2+b+k^2} \sin rvz}{H \cos rvH \left( \frac{dy}{dk} \right)} \quad (x > 0) \]

\[ = -a^2 \int_0^\infty \frac{\sin rvz}{\sin rvH} e^{ikz} \ dk \]

\[ - \frac{\pi i a^2 e^{-k^2+b+k^2} \sin rvz}{H \cos rvH \left( \frac{dy}{dk} \right)} \quad (x < 0) \]

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