1. Introduction.

Recently there has been a renewal of interest in the problem of supplying a rational basis and suitable extensions of Saint-Venant's principle.

Two are the main motivations of this renewal of interest:

1) the need for specifying the mechanical assumptions that allow an extension of the principle to bodies of general shape;
2) the possibility of using the a priori bound techniques of modern analysis to describe the behaviour of the solution of an elastostatic problem.

In 1885 Boussinesq [1], trying to give a general statement of Saint-Venant's principle, made a conjecture on extinction properties of the solution as the loaded region shrinks. His conjecture was criticized by v. Mises [5] who proved it false producing an explicit counterexample and made a revised conjecture. V. Mises' conjecture was proved true later by Sternberg [6]. In a different approach to a rational extension of the principle a given system of surface loads is considered, and the behaviour of a suitable measure of the solution with the distance from the loaded region is studied. Along these guidelines Zanaboni [2], [3], [4] proposed to choose, as an extinction measure, the strain energy of that part of the body beyond a certain distance from the loaded region. The same approach was followed by Toupin [8], who found an upper bound for the strain energy in a semi-infinite cylinder loaded on its end face, and by Knowles [9], who proved a similar upper bound for the plane problem.1

In the present paper we obtain a result similar to Toupin's and Knowles', but for bodies of general shape. We make use of techniques similar to those introduced by Bramble and Payne [7] in order to prove some a priori bounds for the solution of elastostatic problems with various boundary conditions. A thorough discussion of these techniques will be found in a forthcoming book by Villaggio [11] on qualitative methods in elasticity.

2. Formulation of the problem.

Let \( \mathcal{B} \) be a bounded region, regular in the sense of Kellogg (v. Gurtin [10], Sect. 5), filled with an elastic homogeneous and isotropic material.

The constitutive equation is then

\[
S = C \{E\} = 2\mu E + \lambda (\text{tr } E) \mathbf{1}
\]

here \( S \) is the stress tensor; \( E \) is the infinitesimal strain tensor, expressed in terms of the displacement vector \( u \) as

\[
2E = \nabla u + (\nabla u)^T;
\]

\( C \) is the fourth order tensor of elasticity; \( \lambda, \mu \) are the Lamé moduli.

Let \( \pi \) be a plane, dividing the body in two disjoint parts \( \mathcal{B}_- \) and \( \mathcal{B}_0 \) (v. fig. 1) so that

\[
\mathcal{B}_- \cup \mathcal{B}_0 = \mathcal{B}, \quad \pi \cap \mathcal{B}_- = \phi.
\]

(1) A discussion of the works quoted above can be found in Gurtin's [10] treatise on linear elasticity.
Let an orthogonal Cartesian system of co-ordinates \((x_1, x_2, x_3)\) be given, with \(x_1\) normal to \(\pi\) and penetrating \(\mathcal{B}_0\), and \(x_2, x_3\) lying in \(\pi\).

We denote by
\[
\mathcal{L}_1 = \{ x \in \mathcal{B} \mid x_1 = l \}
\] (2.1)
the intersection of \(\mathcal{B}\) with the plane parallel to \(\pi\) and at a distance \(l\) from it, (we assume \(\mathcal{L}_1\) to be simply connected for every \(l\)); we further denote by
\[
\mathcal{B}_1 = \{ x \in \mathcal{B} \mid x_1 \geq l \}
\] (2.2)
the part of \(\mathcal{B}\) beyond \(l\).

We shall consider the following traction problem on \(\mathcal{B}\):
\[
\begin{align*}
\text{div } S &= 0 \quad \text{on } \mathcal{B} \\
S n &= 0 \quad \text{on } \partial \mathcal{B}_0 = \partial \mathcal{B} \cap \mathcal{B}_0 \\
S n &= \hat{s} \quad \text{on } \partial \mathcal{B}_- = \partial \mathcal{B} \cap \mathcal{B}_-
\end{align*}
\] (2.3) (2.4) (2.5)
under the equilibrium conditions
\[
\begin{align*}
&\int_{\partial \mathcal{B}_-} \hat{s} \, da = 0, \\
&\int_{\partial \mathcal{B}_-} x \times \hat{s} \, da = 0
\end{align*}
\] (2.6)
which are necessary for a correct formulation of the problem.\(^{3}\) In particular, we shall take the behaviour of the functional
\[
\mathcal{W}(l) = \int_{\mathcal{B}_1} S \cdot E \, dv.
\] (2.7)
as a measure of the behaviour of the solution in the region \(\mathcal{B}_0\).\(^{3}\) Here \(S\) and \(E\) denote the strain and stress fields corresponding to the solution of the traction problem formulated above. The use of studying the behaviour of a functional of this type lies in the possibility of deriving a local bound for the stress tensor from an estimate of the strain energy.

We now obtain a formula to be used later. In view of the reduction formulae for triple integrals we write
\[
\mathcal{W}(\lambda) = \int_{\mathcal{B}_1} S \cdot E \, dv = \int_{\mathcal{B}_1} \int_{X_1} S \cdot E \, dx_1, \quad (2.8)
\]
where
\[
\lambda = \max_{x \in \mathcal{B}_1} x_1.
\]
Differentiating (2.8) with respect to \(\lambda\) yields
\[
\mathcal{W}'(\lambda) = -\int_{X_1} S \cdot E \, da. \quad (2.9)
\]

3. An a priori bound for the strain energy.

We derive here an a priori bound for the strain energy of \(\mathcal{B}_1\) in order to establish the main result of the paper.\(^{4}\)

Let \(u, E, S\) be the solution of the traction problem formulated on \(\mathcal{B}_0\), with zero body forces and surface forces that are non zero only on \(\mathcal{L}_1\).

Let \(v\) be the displacement field
\[
v = u + v_0 + \omega_0 \times x
\]
where \(v_0\) and \(\omega_0\) are chosen so that
\[
\int_{\mathcal{L}_1} v \, da = 0, \quad \int_{\mathcal{L}_1} x \times v \, da = 0.
\] (3.1)

Let \(\bar{E}\) and \(\bar{S}\) be the strain and stress fields corresponding to \(v\); since \(u\) and \(v\) differ only by a rigid body motion, we have
\[
\bar{E} = E, \quad \bar{S} = S
\] (3.2)

Using the symmetry of the stress tensor and the divergence theorem we write
\[
\begin{align*}
&\int_{\mathcal{B}_1} \bar{S} \cdot \bar{E} \, dv = \int_{\mathcal{B}_1} \bar{S} \cdot \nabla v \, dv = \int_{\mathcal{B}_1} \text{div}(\bar{S}v) \, dv - \\
&\quad - \int_{\mathcal{B}_1} \text{div} \bar{S} \cdot v \, dv,
\end{align*}
\] (3.3)
and we conclude from (3.2), (3.3) and the divergence theorem that
\[
\begin{align*}
&\int_{\mathcal{B}_1} S \cdot E \, dv = \int_{\mathcal{B}_1} \bar{S} \cdot \bar{E} \, dv = \int_{\mathcal{B}_1} S v \cdot n \, da - \\
&\quad - \int_{\mathcal{B}_1} \text{div} S \cdot v \, dv.
\end{align*}
\] (3.4)

Using again the symmetry of the stress tensor and observing that, by (2.3), (2.4),
\[
\text{div } S = 0 \quad \text{on } \mathcal{B}_1 \quad \text{and } \quad S n = 0 \quad \text{on } \partial \mathcal{B}_1 - \mathcal{L}_1
\]
we rewrite (3.4) as
\[
\int_{\mathcal{B}_1} S \cdot E \, dv = \int_{\mathcal{B}_1} S n \cdot v \, da
\] (3.5)
and make use of Cauchy-Schwarz inequality to get
\[
\int_{\mathcal{B}_1} S \cdot E \, dv \leq \left( \int_{\mathcal{B}_1} |S n|^2 \, da \right)^{1/2} \left( \int_{\mathcal{B}_1} v^2 \, da \right)^{1/2}
\] (3.6)

By Cauchy-Schwarz inequality and since \(|n| = 1\) and \(C\) is symmetric, we obtain further
\[
|S n|^2 \leq |S|^2 = C |E| \cdot C |E| = E \cdot C^T E.
\] (3.7)

\(^{(2)}\) This problem is the natural three dimensional version of the plane problem studied by Knowles [9].

\(^{(2)}\) The introduction of a functional of this kind is due to Zanaboni [2], [3], [4].