Orthogonal rational functions and quadrature on the unit circle

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In this paper we shall be concerned with the problem of approximating the integral
\[ I_\mu(f) = \int_{-\pi}^{\pi} f(e^{i\theta}) \, d\mu(\theta), \]
by means of the formula
\[ I_{n\mu}(f) = \sum_{j=1}^{n} A_j(n)f(x_j(n)) \]
where \( \mu \) is some finite positive measure. We want the approximation to be so that \( I_\mu(f) = I_{n\mu}(f) \) for \( f \) belonging to certain classes of rational functions with prescribed poles which generalize in a certain sense the space of polynomials. In order to get nodes \( \{x_j(n)\} \) of modulus 1 and positive weights \( A_j(n) \), it will be fundamental to use rational functions orthogonal on the unit circle analogous to Szegő polynomials.

1. Introduction

In this paper, we are concerned with complex function theory on the unit circle. We start with the introduction of some notation for the unit circle, the open unit disc and the exterior of the unit circle
\[ T = \{z : |z| = 1\}, \quad D = \{z : |z| < 1\}; \quad E = \{z : |z| > 1\}. \]
For a given sequence \( \{\alpha_j\}_{j=1}^{\infty} \subset D \), we consider for \( n = 0, 1, \ldots \) the nested space \( \mathcal{L}_n \) of rational functions of degree \( n \) at most which are spanned by the basis of

\[ \text{The work of the first author is partially supported by a research grant from the Belgian National Fund for Scientific Research.} \]
partial Blaschke products \( \{B_k\}_{k=0}^n \) where \( B_0 = 1, \ B_n = B_{n-1} \zeta_n \) for \( n = 1, 2, \ldots \) and the Blaschke factors are defined as

\[
\zeta_n(z) = \frac{\bar{\alpha}_n}{|\alpha_n|} \frac{\alpha_n - z}{1 - \bar{\alpha}_n z}.
\]

By convention, we set \( \bar{\alpha}_n/|\alpha_n| = -1 \) for \( \alpha_n = 0 \). Sometimes, we shall also write

\[
B_n(z) = \eta_n(z) \frac{\omega_n(z)}{\pi_n(z)}; \quad \eta_n = (-1)^n \prod_{j=1}^n \frac{\bar{\alpha}_j}{|\alpha_j|};
\]

\[
\omega_n(z) = \prod_{j=1}^n (z - \alpha_j) \quad \text{and} \quad \pi_n(z) = \prod_{j=1}^n (1 - \bar{\alpha}_j z).
\]

These spaces of rationals have been studied in connection with the Pick–Nevanlinna problem \([8–10]\) and in many applications \([1,2,5]\). Note that if all the \( \alpha_i \) are equal to zero, the spaces \( \mathcal{L}_n \) collapse to the spaces \( \Pi_n \) of polynomials of degree \( n \). Clearly \( \mathcal{L}_n \) is a space of rational functions with prescribed poles \( 1/\alpha_i; \ i = 1, 2, \ldots, n \) which are all in \( \mathbb{E} \), that is,

\[
\mathcal{L}_n = \text{span}\{B_k; \ k = 0, 1, \ldots, n\} = \left\{ \frac{p_n}{\pi_n}; \ p_n \in \Pi_n \right\}.
\]

We also introduce the following transformation \( f_\ast(z) = \bar{f}(1/\bar{z}) \) (\( f_\ast(z) = \bar{f}(z) \) on \( \mathbb{T} \)), which allows to define for \( f_n \in \mathcal{L}_n \) the superstar conjugate as

\[
f_n^\ast(z) = B_n(z)f_n(z).
\]

Note that in the polynomial case, i.e., when \( \alpha_i = 0; \ i = 0, \ldots, n \), hence \( B_n(z) = z^n \), the coefficient \( p^\ast(0) \) is the leading coefficient. In analogy, we shall call here \( f_n^\ast(\alpha_n) \) the leading coefficient of \( f_n \in \mathcal{L}_n \) (with respect to the basis \( \{B_k\} \)).

Next we consider a positive measure \( \mu \) on the unit circle and orthonormalize the basis for \( \mathcal{L}_n \) with respect to the inner product

\[
\langle f, g \rangle = \int_{-\pi}^{\pi} f(e^{i\theta})g(e^{i\theta}) \, d\mu(\theta)
\]

to generate an orthonormal system \( \phi_0 = 1, \ \phi_k \in \mathcal{L}_k - \mathcal{L}_{k-1}, \ \phi_k \perp \mathcal{L}_{k-1}, \ k = 1, 2, \ldots, n \). They are uniquely defined if we require that their leading coefficient \( \kappa_n = \phi_n^\ast(\alpha_n) \) is positive. In \([2]\) it was proved that all the zeros of \( \phi_n^\ast(z) \) are in \( \mathbb{E} \) and that it satisfies the orthogonality property \( \phi_n^\ast \perp \zeta_n \mathcal{L}_{n-1}, \ n = 1, 2, \ldots \) where

\[
\zeta_n = \mathcal{L}_{n+1} = \left\{ \left. f \in \mathcal{L}_n : f(\alpha_n) = 0 \right\} = \mathcal{L}_n(\alpha_n).
\]

(Obviously, the zeros of \( \phi_n \) lie in \( \mathbb{D} \).)

Other bases can be used for \( \mathcal{L}_n \) (see \([2,6]\)). In particular, the following will be of great interest for our purposes. If we set \( V_0 = 1, \ V_k = B_{k-1}/(1 - \bar{\alpha}_k z); \ k = 1, 2, \ldots, \) then in \([2]\) it was proved that \( \mathcal{L}_n = \text{span}\{1, (z - w)V_1, (z - w)V_2, \ldots, (z - w)V_n\}, \ w \) being any complex number.