On a regular form defined by a pseudo-function

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We study forms which are closely related to the Legendre form. The Stieltjes functions and the recurrence coefficients of the polynomials associated with these forms are explicitly given.

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0. Introduction

Recently, we have given a representation for the inverse of Tchebychev forms [8]

\[
\left\langle T^{-1}, f \right\rangle = \frac{1}{\pi} \int_{-1}^{1} \frac{f(x)}{1 + \sqrt{1 - x^2}} \, dx - \frac{1}{\pi} \left\langle Pf \frac{Y(1 - x^2)}{x^2}, f \right\rangle,
\]

where

\[
\left\langle T, f \right\rangle = \frac{1}{\pi} \int_{-1}^{1} \frac{f(x)}{1 + \sqrt{1 - x^2}} \, dx
\]

are respectively the Tchebychev forms of second and first kind; and

\[
\left\langle Pf \frac{Y(1 - x^2)}{x^2}, f \right\rangle = Pf \int_{-\infty}^{+\infty} \frac{Y(1 - x^2)}{x^2} f(x) \, dx
\]

\[
= \lim_{\varepsilon \to 0^+} \left( \int_{-1}^{-\varepsilon} \frac{f(x)}{x^2} \, dx + \int_{-\varepsilon}^{+1} \frac{f(x)}{x^2} \, dx - \frac{2}{\varepsilon} f(0) \right)
\]

for each polynomial $f$ and with

\[
Y(x) = \begin{cases} 
0, & x \leq 0, \\
1, & x > 0.
\end{cases}
\]
The inverse \( u^{-1} \) of \( u \) is defined by \( u^{-1}u = uu^{-1} = \delta \) where \( \langle \delta, f \rangle = f(0) \) and the product \( uv \) of the two forms is given by the moments \( (uv)_n := \langle uv, x^n \rangle = \Sigma_{\mu + \nu = n}(u)_{\mu}(v)_{\nu}, n \geq 0 \) [5,6].

Let us recall the definition of regularity: a form \( u \) is called regular if there exists a monic polynomial sequence \( \{P_n\}_{n \geq 0} \) such that
\[
\langle u, P_n P_m \rangle = k_n \delta_{n,m}, \quad n, m \geq 0; \quad k_n \neq 0, \quad n \geq 0.
\]

Our aim is to prove the regularity of the form \( Pf Y(1 - x^2)/x^2 \) and to build the orthogonal sequence \( \{Z_n\}_{n \geq 0} \)
\[
\left\langle Pf \frac{Y(1 - x^2)}{x^2}, Z_n Z_m \right\rangle = k_n \delta_{n,m}, \quad n, m \geq 0
\]
with \( k_n \neq 0, n \geq 0 \) and
\[
Z_0(x) = 1, \quad Z_1(x) = x
\]
\[
Z_{n+2}(x) = xZ_{n+1}(x) - \gamma_{n+1}Z_n(x), \quad n \geq 0.
\]

For other examples of generalized weights, see [2–4].

1. The form \( Pf[V(x)]/x^2 \)

More generally, let us consider the form \( Pf [V(x)]/x^2 \) where we suppose that \( V \) is a locally integrable function with rapid decay satisfying
\[
Pf \int_{-\infty}^{+\infty} \frac{V(x)}{x^2} \, dx \neq 0, \quad \int_{-\infty}^{+\infty} V(x) \, dx = 1.
\]
Let \( v \) be defined by \( \langle v, f \rangle := \int_{-\infty}^{+\infty} V(x)f(x) \, dx \). Further, we suppose that \( v \) is regular. Then, let \( u \)
\[
\langle u, f \rangle = \lambda Pf \int_{-\infty}^{+\infty} \frac{V(x)}{x^2} f(x) \, dx \tag{1.1}
\]
with
\[
\lambda = \left( Pf \int_{-\infty}^{+\infty} \frac{V(x)}{x^2} \, dx \right)^{-1}.
\]
We have
\[
x^2u = \lambda v, \quad \lambda = (u)_2. \tag{1.2}
\]
Equivalently [6]
\[
u = \delta - (u)_1 D\delta + \lambda x^{-2}v, \tag{1.3}
\]
where the left-multiplication \( hw \) is defined by \( \langle hw, f \rangle = \langle w, hf \rangle \), the derivative