The semilocal convergence of a generalization of Brett’s and Brown’s methods

Zhijian Huang *

Department of Statistics and Operations Research, Fudan University, Shanghai 200433, P.R. China

Received 21 November 1991; revised 15 April 1993
Communicated by C. Brezinski

In this paper, we discuss the semilocal convergence of Martínez’s generalization of Brent’s and Brown’s methods. Through a careful investigation of the algorithm structure, we convert Martínez’s generalized method into an approximate Newton method with a special error term. Based on such equivalent variation, we prove the semilocal convergence theorem of Martínez’s generalized method. This is a complementary result to the convergence theory of Martínez’s generalized method.

Keywords: Brett’s method, Brown’s method, Martínez’s generalized method, approximate Newton method, semilocal convergence theorem.

1. Introduction

In [2,3], Brown presented a discretized method, which was improved by Gay [5] afterwards, for solving systems of nonlinear equations

\[ F(x) = 0, \quad F : D \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^n, \]

where \( F(x) = [f_1(x), \ldots, f_n(x)]^T \). This method has a quadratic convergence rate and requires only \( (n^2 + 3n)/2 \) function evaluations per iteration. Based on the idea of the orthogonal triangular factorization, Brent [1] proposed an efficient method with the same convergence property and number of function evaluations as Brown’s method. However, as observed by Cosnard [4] and Moré and Cosnard [8], Brent’s method is numerically more stable than Brown’s method. In [6], Martínez presented a generalized version of Brent’s and Brown’s methods. In Martínez’s generalized method, the set of functions \( f_1, \ldots, f_n \) is divided into \( N \) blocks so that the functions of the \( j \)-th block are evaluated at the same \( n - p + 1 \) points, where \( p \) is the

* Current address: Department of Mathematics, University of Bergamo, Piazza Rosate 2, 24100 Bergamo, Italy.
number of functions in block 1 through \( j - 1 \). This class of methods includes the discretized Newton method and Brent's and Brown's methods. Local convergence properties and some numerical experience are given in [6] and [7].

In this paper, we discuss the semilocal convergence of Martínez's generalization of Brent's and Brown's methods. Because of the complexity of the algorithm structure, the traditional major function approach for proving semilocal convergence cannot be applied. Through a careful investigation of the algorithm structure, we convert Martínez's generalized method into an approximate Newton method with a special error term. Based on such equivalent variation, we prove the semilocal convergence theorem of Martínez's generalized method, in which the convergence conditions mainly depend on the behavior of the mapping at the initial point. This is a complementary result to the convergence theory of Martínez's generalized method.

2. Description of the algorithm

In this paper, we take the Frobenius norm for matrices and the Euclidean norm for vectors.

Let \( \mathcal{F} = \mathcal{F}(K_1, K_2) \) be the set of matrices of order less than or equal to \( n \) such that for all \( A \in \mathcal{F} \), \( A^{-1} \) exists, \( \|A\| \leq K_1 \) and \( \|A^{-1}\| \leq K_2 \). Let \( N \) be a positive integer less than or equal to \( n \) and \( q_i \) be a positive integer for all \( i = 1, \ldots, N \) such that \( \sum q_i = n \). Suppose that \( B_k \) is an \( n \times n \) matrix belonging to \( \mathcal{F} \), \( h_k \neq 0 \). Then, the algorithm procedure from \( x^{(k)} \) to \( x^{(k+1)} \) can be described as follows:

**ALGORITHM (GBBM)**

**Step 1.** \( A_1^{(k)} = B_k, y_1^{(k)} = x^{(k)} \).

**Step 2.** For \( j = 1, 2, \ldots, N \), do steps 3 to 5.

**Step 3.** Define

\[
p_{j+1} = \sum_{i=1}^{j} q_i \quad (p_1 = 0),
\]

\[
A_j^{(k)} = (C_1^{(j)}, \ldots, C_n^{(j)}).
\]

Compute

\[
\begin{align*}
(a_j^{(k)})_{rt} &= [f_{p_j+r}(y_j^{(k)} + h_k C_t^{(j)}) - f_{p_j+r}(y_j^{(k)})]/h_k \\
&\quad \text{for } t = p_j + 1, \ldots, n; r = 1, \ldots, q_j \\
(a_j^{(k)})_{rt} &= 0, \quad \text{for } t = 1, \ldots, p_j; r = 1, \ldots, q_j
\end{align*}
\]  

(2.1)

**Step 4** Find an \((n - p_j) \times (n - p_j)\) matrix \( \bar{U}_j^{(k)} \) such that for