NUMERICAL METHOD FOR EVALUATING THE RESPONSE OF ROTATING SHAFTS TO EXCITATION
Gianfranco Capriz *

SUMMARY: To obtain an accurate forecast of the behaviour of a shaft rotating on lubricated bearings under excitation, an adequate model must be introduced to represent the response of the bearings; a linear model of the response is sufficient for many practical purposes, but allowance must be made for the variation of film stiffness, cross-stiffness, etc., with direction. As a consequence the shaft must be assumed to move in an elliptical, rather than circular, whirl. The phenomenon which must be tackled is thus more complex than that envisaged in König's study, for instance, see Ref. [2]. We show here nevertheless that it is possible to devise an appropriate generalization of the Myklestad-Holzer method, so that the problem can be solved by matrix manipulations.

1. Introduction.

The investigation was prompted by the following considerations: I) To base the design of a shaft largely on the calculation of its critical speeds is at times misleading since damping may allow the running of a well-balanced shaft close to or even at a critical speed. II) It is of interest to have means of predicting the behaviour of a shaft under accidental unbalance, such as may be caused by the stripping of some blades in a turbine. III) The response of a shaft to disturbance is greatly influenced by the characteristics of its bearings: hence a careful representation of these is important. The peculiar properties of fluid films usually lead to elliptic motion.

The paper is divided into sections. Sects. 2, 3, 4 are devoted to establishing the basic equations; a method (of the Myklestad-Holzer type; see, for instance, Ref. [1]) for the numerical solution of the problem is then described, having in mind, of course, the use of a computer. In Sect. 6 the problems arising from bearings effects are tackled.

To achieve compactness many details (which are called for to plan a flexible computer programme) are left out of this paper: for instance distributed damping and magnetic forces are not mentioned here. We refer to the internal report of Ref. [9] for these and other questions.

2. Dynamic equations.

The analysis is based on the simple theory of beams (Refs. [3], Ch. XVIII; [1], Ch. IV; [4], Chs. V, VI) with the additional consideration of shear and gyroscopic effects (Ref. [4], nos. 48, 55; Ref. [5]). However, a brief restatement of the principles involved is in order, as it gives the opportunity for precise definitions.

A fixed system of reference $S^*$ (origin $O_1$; unit axial vectors $e_1$, $e_2$, $e_3$) is chosen so that the third axis (axis of $e_3$) goes through the line of centres of two bearings, but either the line of the elastic centres $C$ or the line of the centres of mass $G$ (of elementary slices) may be different from this axis even in the undisturbed state (in the first case due to a permanent bend, in the second due to lack of balance). Because of this we introduce the notation $C_s, G_s$ for the positions of $C$ and $G$ in the undisturbed state.

To achieve compactness many details (which are called for to plan a flexible computer programme) are left out of this paper: for instance distributed damping and magnetic forces are not mentioned here. We refer to the internal report of Ref. [9] for these and other questions.

To calculate correctly the elastic forces and moments arising during whirl the displacement $w(z, t)$ must be interpreted as the displacement at the instant $t$ of the point $C$ (relating to the cross-section of coordinate $z$). At the same time a local measure of unbalance is given by the vector $e(z)$ which joins $O(z)$ (the point of the cross-section of coordinate $z$) on the axis) to $G_s(z)$. If $w_0(z)$ is the displacement due to permanent bending, then:

$$e(z) = w_0(z) + C_s G_s.$$

Fig. 1. The shaft in the undisturbed state, showing (grossly exaggerated) initial bend.

* Professore di Meccanica Razionale, Centro Studi Calcolatrici Elettroniche, Università di Pisa.
A second system of reference \( s(\Omega, i_1, i_2, i_3 = c_3) \) is also introduced, which rotates with the shaft at the same steady speed \( \omega \) (radians per unit time) around the third axis.

Calling the mass \( M \), the resultant external force \( R^{(e)} \) and \( M^{(m)} \) the rate of change of moment of momentum (per unit length of the shaft), the equations of motion of the shaft are:

\[
\frac{\partial S}{\partial \xi} = \frac{\mu A_c}{g} \frac{\partial^2 w}{\partial t^2} - R^{(e)},
\]

\[
\frac{\partial M}{\partial \xi} = S \times c_3 + M^{(m)},
\]

where \( S \) is the shear force and \( M \) the bending moment.

The external forces contributing to \( R^{(e)} \) are unbalance, damping and, in rotors of electrical machines, magnetic forces. For simplicity, only the first source of excitation is considered here.

The vector \( e \) is usually known through its components on the moving frame:

\[
e = e_1 i_1 + e_2 i_2,
\]

and we will express forces due to lack of balance in terms of \( e_1, e_2 \):

\[
R^{(e)} = \frac{\mu A_c}{g} \omega^2 (e_1 i_1 + e_2 i_2).
\]

An explicit expression for \( M^{(m)} \) can be based on the observation that the motion of any elementary slice of the shaft is approximately rigid, with angular velocity \( \Omega(t_1 + t_2 + t_3) \), if the vector \( p = p_1 i_1 + p_2 i_2 \) is defined through the formula:

\[
\frac{\partial}{\partial \xi} \left( \frac{\partial (w_0 + w)}{\partial t} \right) = p \times c_3,
\]

and \( p_3 = \omega \).

Using the notation \( j_1 d\xi, j_2 d\xi, j_3 d\xi (j_1 = j_2 = j) \) for the principal moments of inertia of a rigid elementary slice of the shaft of thickness \( d\xi \) (calculated as though there was no permanent bending) we have then for \( M^{(m)} \):

\[
M^{(m)} = j_3 \left[ \frac{\partial p_3}{\partial t} - (j_1 + j_2) p_1 i_1 + j_3 p_3 i_3 \right] i_2;
\]

where the indices must be interpreted cyclically.

Naturally throughout these formulæ second order terms are dropped, which would be required to account for lack of exact parallelism between the principal axes of inertia of the slice during motion and the unit vectors \( i_1, i_2, i_3 \).

From (2.4), calling \( \theta \) the slope, we get:

\[
M^{(m)} = j_3 \omega \frac{\partial \theta}{\partial t} + j c_3 \times \frac{\partial^2 \theta}{\partial t^2}.
\]

But during a circular whirl of steady angular speed \( r c_3 \),

\[
\frac{\partial \theta}{\partial t} = r c_3 \times \theta,
\]

\[
\frac{\partial^2 \theta}{\partial t^2} = -r^2 \theta,
\]

hence

\[
M^{(m)} = j_3 \omega r c_3 \times \theta + \omega^2 \frac{\partial \theta}{\partial t} - j c_3 \times \frac{\partial \theta}{\partial t}.
\]

in our case, of course, \( \omega = v \).

3. Dependence of displacements on forces and moments. Boundary conditions.

When it is deemed convenient to account for the direct effect of shear force on the displacement in a bar, the usual relationship between bending moment and curvature of the displaced centre line is complemented by the following assumption: the displacement \( w \) can be split into the sum of two components \( w_*, w_{**} \), of which the first obeys the usual relationship:

\[
M = E I c_3 \times \frac{\partial^2 w_*}{\partial \xi^2}.
\]

(\( E \), modulus of elasticity; \( I \), second moment of area of cross-section), whereas the derivative of the second is proportional to the shear force

\[
S = \frac{A_0 B}{k} \frac{\partial w_{**}}{\partial \xi}.
\]

(\( B \), shear modulus; \( A_0 \), area of cross-section; \( k \), Timoshenko's constant).

Eqs. (3.1), (3.2) imply that \( M \) and \( w \) can be put into a direct relationship:

\[
M = E I c_3 \times \frac{\partial^2 w_*}{\partial \xi^2} - \frac{k}{B A_0} S \left( \frac{\partial}{\partial \xi} \left( \frac{k}{B A_0} S \right) \right).
\]

and there is no need to make further reference to \( w_*, w_{**} \); we can use Eq. (3.3) and \( w \) only.

Eqs. (3.1), (3.2) justify, however, a special notation, \( \theta \), for the slope due to bending

\[
\theta = \frac{\partial w}{\partial \xi} - \frac{k}{A_0 B} S.
\]

No essential change would actually occur in the following analysis, if \( \theta \) were taken to be equal simply to \( \partial w/\partial \xi \); but with the notation (3.4) some simplification is achieved. Note also that rigour requires that \( w_* \) rather than \( w \) enters into Eq. (2.3), and although gyroscopic and shear actions are normally small so that there is no strict need to account for compound effects, we will now use the formula:

\[
\frac{\partial (\theta + \theta_0)}{\partial t} = p \times c_3,
\]

rather than (2.3), to define \( p_1 \) and \( p_2 \).

The indefinite equations (2.1), (2.2), (2.4), (3.4), (3.5), must be associated with appropriate boundary conditions (at each end of the shaft) and with transition conditions (at the bearings).

If the shaft runs on plain lubricated bearings, fluid film effects are of great importance. Within the limits of our analysis they must be thought of as linear effects.