A Linear Heat Problem with a Moving Interface

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1. Introduction

In this paper we consider a one-dimensional heat conduction problem which possesses some unusual features, and which may be formulated as follows: we seek functions \( u_1(x, t) \) and \( u_2(x, t) \) governed by the equations

\[ a_1^2 \frac{\partial^2 u_1}{\partial x^2} = \frac{\partial u_1}{\partial t}, \quad 0 < x < s(t); \]
\[ a_2^2 \frac{\partial^2 u_2}{\partial x^2} = \frac{\partial u_2}{\partial t}, \quad s(t) < x < \infty, \]

and satisfying the initial and boundary conditions

\[ u_1(0, t) = g(t), \quad t > 0 \]
\[ u_1[s(t), t] = u_2[s(t), t] = h(t), \quad t > 0 \]
\[ u_2(x, 0) = f(x), \]

where the functions \( g(t), h(t) \) and \( f(x) \) are given. The motion of the boundary \( x = s(t) \) separating the two regions is also specified, and we suppose furthermore that \( s(0) = 0 \). The problem as stated above for the semi-infinite domain \( x > 0 \) consists in reality of two distinct problems, one relating to the interval \( 0 < x < s(t) \) (region 1) and the other to the interval \( x > s(t) \) (region 2). Problems of this type involving one or other region arise in a variety of physical situations some of which have been discussed by authors \([1 - 4]^3\). In these papers solutions in closed form are given to examples in which \( g(t) \) and \( h(t) \) are constants, \( f(x) \) is a linear function, and \( s(t) \) is given by

\[ s = \alpha t \]
\[ s = \beta t^{1/2}, \]

but, save in one instance \([3]\), the techniques adopted cannot readily be extended to resolve problems involving boundary conditions more general than these.

1) Imperial College, University of London.
2) Numbers in brackets refer to References, page 206.
We shall shew that for both regions the determination of a solution can be made to depend upon the possibility of solving a singular Fredholm equation of the first kind. In particular, when the motion of the boundary is restricted to the form (6) or (7) the kernel of the integral equation reduces to a type which enables a formal solution to be obtained in terms of a contour integral. We find that when the boundary moves with uniform speed the conditions for existence of the integral are not unduly restrictive, but when it moves according to (7) conditions must be imposed which are unlikely to be satisfied in many practical problems. For this latter case an alternative form of solution is discussed.

2. The Temperature Distribution in Region 1

2.1 Derivation of the Integral Equation

In region 1 the problem reduces to the determination of the temperature field $u(x, t)$ satisfying

$$a^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}, \quad 0 < x < s(t),$$

(8)

and

$$u(0, t) = g(t), \quad t > 0,$$

(9)

$$u[s(t), t] = h(t), \quad t > 0,$$

(10)

where, for convenience, the suffix has been dropped.

It is evident from condition (9) that some simplification results by introducing a doublet of strength $g$ at the origin, and taking a new dependent variable

$$v = u - \frac{x}{2a\sqrt{\pi}} \int_0^t g(\tau) G(x, t - \tau) (t - \tau)^{-1} d\tau,$$

(11)

where

$$G(x, t) = t^{-1/2} \exp\left(-\frac{x^2}{4a^2 t}\right).$$

Since

$$u(0, t) = v(0, t) + g(t)$$

it follows that $v$ must satisfy

$$a^2 \frac{\partial^2 v}{\partial x^2} = \frac{\partial v}{\partial t}, \quad 0 < x < s(t),$$

(12)

$$v(0, t) = 0, \quad t > 0$$

(13)

and

$$v[s(t), t] = h(t) - \frac{s(t)}{2a\sqrt{\pi}} \int_0^t g(\tau) G[s(t), t - \tau] (t - \tau)^{-1} d\tau.$$

(14)