On Confinement of Fermions in Strongly Coupled Lattice Gauge Theory

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Abstract. A lattice theory of Fermi fields of mass \( m \) coupled to gauge fields in the region where \( m \) and the gauge field coupling constant \( g \) are large is studied. It is shown that the energy of some states composed of a fermion and a distant antifermion with a string in between grows at least linearly with the distance if \( 1 < g^6 < m < g^{10g} \).

1. Introduction

The lattice gauge field theory was formulated by Wilson [15] with the hope that the mechanisms behind the long distance behavior of the continuum fields could be understood in the simplified lattice models, see also [1, 7]. The primary aim was to understand the quark confinement. Working with the QED Wilson formulated a criterion for charge confinement which involved only the electromagnetic field (no Fermi fields):

\[
\exp\left(\int_S A_\mu dx^\mu\right) \sim \exp(-C(s)) \quad \text{and} \quad C(s) \text{ is proportional to the area of the two-dimensional cube } s
\]

then charge should be confined; if \( C(s) \) is proportional to the circumference of \( s \) then no confinement occurs.

The criterion was based on the analysis of the expansion of Euclidean propagators of full lattice QED into powers of, say, inverse fermion mass, interpreted as a sum over fermion-antifermion trajectories. Each path \( \sigma \) contributed a lattice version of \( \exp\left(\int_\sigma \left[ i e A_\mu dx^\mu\right]\right) \). It was argued that in the case of the "area law", paths with fermion and antifermion well separated hardly entered. Wilson suggested that in the lattice QED the area law should hold for large coupling constant \( g \). This was confirmed by the rigorous result of Osterwalder-Seiler [10] obtained for a wide class of lattice gauge theories. In the meantime

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other possible mechanisms leading to the area law have been proposed, believed to work for weak or intermediate coupling (instantons, merons) [2, 3, 6, 12, 13].

In the present paper we consider a gauge field interacting with a color multiplet of Dirac Fermi fields (in Euclidean region). Our result is a rigorous step in proving occurrence of confinement for large Dirac fields mass $m$ and large coupling constant $g$, starting from a different criterion, more appealing to the physical situation. It may be also viewed however as a rigorous step on the way to substantiate the Wilson’s criterion.

The criterion we use is based on the following rough idea: confinement occurs if the physical (i.e. gauge invariant!) states with gauge charges concentrated in well separated regions have big energy, growing with the separation. This idea of confinement, where the use of gauge invariant states is a crucial point, is very close to the one used in [7]. Our test-states $X_r$ consist of a fermion and an antifermion connected by a gauge field string (necessary to have gauge invariance). We are able to prove that once the spins of the fermion and the antifermion are correlated in a certain way the energy of the state grows at least linearly in $r$ for some $m \gg g \gg 1$. The correlation between spins of the pair seems to result only from our inability to dismiss this assumption.

The basic technical tool we use is the cluster expansion—a generalization of the one worked out in [9, 10] for pure lattice Yang-Mills theory (without fermions). Section 2 is devoted to the formulation and to the proof of convergence of the cluster expansion. As usually, once the convergence is proven, existence of the exponentially clustering infinite volume theory follows and the standard construction [11] gives the physical Hilbert space and the transfer matrix $e^{-2H}$, since the Osterwalder-Schrader positivity holds, as proven in [10].

The confinement bound is deduced from the estimate

$$
\frac{1}{\|X_r\|^2} \langle X_r | e^{-2H} X_r \rangle \leq e^{-0(1)r}. \tag{1}
$$

To obtain (1) we bound $\langle X_r | e^{-2H} X_r \rangle$ from above by $e^{-C_1 r}$ directly from the cluster expansion. The missing lower bound on $\|X_r\|^2$, $\|X_r\|^2 \geq e^{-C_2 r}$, with $C_2 < C_1$, is more difficult. One can bound $\|X_r\|^2$ from below computing it in the lowest order of the strong coupling perturbation calculus and estimating the correction by the cluster expansion, which is well suited for that. However this does not lead to the searched bound directly. Nevertheless this bound can be obtained if we use additionally a convexity of log $\|X_r\|$ (Proposition 2) which can be proven by a sort of Nelson symmetry argument. This is where our nasty assumption on correlation of spins of the fermion and the antifermion enters. The estimates leading to the lower linear bound on the energy of $X_r$ compose Section 3 of the paper.

2. Cluster Expansion

The model we study is essentially the same as the one considered by Osterwalder-Seiler [10, Sections II.2 and II.3]. We shall briefly recall and supplement their notation, introducing some minor changes.