Direct Solution of Partial Difference Equations

ERNST MERZARTH

Received May 5, 1966

Introduction

LYNCH, RICE, and THOMAS describe in [1] a method for the direct and explicit solution of the partial difference equations arising from separable boundary-value problems, i.e. partial difference equations which arise when applying the ordinary difference method. We describe in this paper a method for the direct solution of the partial difference equations which arise when applying the Hermitian difference method (Mehrstellenverfahren) to separable boundary-value problems.

1. Second Order Separable Partial Difference Equations

In this paper we consider partial difference equations derived from separable boundary-value problems.

Definition. A boundary-value problem will be called separable if

a) the region of the problem is a rectangle \( R \) whose sides are parallel to the coordinate axes, and if

b) the differential equation has the form\(^1\):

\[-\Delta u + [a(x) + b(y)]u = f(x, y)\]  \( \text{(1.1)} \)

with \( a(x), b(y) \geq 0 \) on \( R \), and if

c) the boundary conditions, given on the four sides \( \Gamma_i \) \( (i = 1, \ldots, 4) \) of \( R \), have the form:

\[\frac{\partial u}{\partial n} + \alpha_i u + \beta_i = g_i(s) \quad \text{on} \ \Gamma_i \quad i = 1, 2, 3, 4\]

where \( \partial u/\partial n \) is the derivative of \( u(x, y) \) in the direction of the outer boundary normal \( n \) and \( s \) is a boundary parameter. The coefficients \( \alpha_i, \beta_i \) shall be non-negative. Besides these boundary conditions, periodic boundary conditions are also admitted.

\(^1\) The definition of a separable boundary-value problem, used in this paper, is more specific than the usual definition which allows differential equations of the form:

\[-a_2(x) u_{xx} + a_1(x) u_x + a_0(x) u - b_2(y) u_{yy} + b_1(y) u_y + b_0(y) u = f(x, y).\]

We assume in the following that the differential equation has always been transformed to the form (1.1).
In the following we consider especially difference equations derived from separable boundary-value problems with constant coefficients. The following boundary-value problem is an example for separable boundary-value problems with constant coefficients:

\[
\begin{align*}
-\Delta u + cu &= f(x, y) \\
u(0, y) &= g_4(y) \\
u_x(0, y) &= g_3(x) \\
u_x(l_1, y) &= g_5(y) \\
u_y(x, 0) &= g_1(x) \\
u_y(x, l_2) &= g_3(x)
\end{align*}
\]

for \(0 \leq x \leq l_1\) and \(0 \leq y \leq l_2\) \hspace{1cm} (1.2)

The difference equations which we shall consider are derived by using the Hermitian difference method (Mehrstellenverfahren). In the region of the boundary-value problem a rectangular mesh of points is placed. In the case of the problem (1.2) the mesh points are

\[
x_i = 0 + i \cdot h_x \quad \text{and} \quad y_k = 0 + k \cdot h_y
\]

with \(h_x = l_x/m\), \(h_y = l_y/n\) where \(m\) and \(n\) are any positive integers. The function values at mesh points are characterized by corresponding subscripts; for instance, \(u_{i,k}\) denotes \(u(x_i, y_k)\) and \(u_{i,k}\) denotes an approximation to it. For replacing the differential operator \(\Delta u\) in the differential equation, we use the following Hermitian formula:

\[
\frac{1}{12} \begin{bmatrix}
1 & 10 & 1 \\
10 & 100 & 10 \\
1 & 10 & 1
\end{bmatrix} \approx \frac{1}{h_x^2} \begin{bmatrix}
1 & -2 & 1 \\
-20 & 20 & -20 \\
1 & 10 & 1
\end{bmatrix} u_{i,k} + \frac{1}{h_y^2} u_{i,k}.
\]

The obtained difference equations approximate the differential equation to order \(h_x^2 + h_y^2\).

2. Another Way of Writing the Difference Equations

We start with another representation of certain systems of difference equations. Any system of difference equations derived from a separable boundary-value problem with constant coefficients by using the Hermitian difference method can be written in the form

\[
S_y U D_x + \mu D_y U S_x + \gamma S_y U S_x = G
\]

(2.1)

where \(U\) is a rectangular matrix formed of all the unknown function values \(u_{i,k}\) where \(D_x, D_y, S_x, S_y\) are tridiagonal matrices of appropriate order and where \(G\) is a given rectangular matrix of the same order as \(U\). \(\mu\) and \(\gamma\) are constants.

\[\text{We use the notation}\]

\[
\begin{bmatrix}
a & b \\
d & e \\
g & h
\end{bmatrix}
\begin{bmatrix}
v_{i,k} \\
v_{i+1,k}
\end{bmatrix}
= \begin{bmatrix}
a v_{i-1,k-1} + b v_{i,k-1} + c v_{i+1,k-1} \\
+ a v_{i-1,k} + b v_{i,k} + c v_{i+1,k} \\
+ a v_{i-1,k} + b v_{i,k} + c v_{i+1,k}
\end{bmatrix}
\]