The method used for computing the definite integral

\[ J := \int_{a}^{b} f(x) \, dx \]

is based on rational extrapolation proposed in [2], which is a variant of the so-called Romberg-integration (see [1, 5]). This method runs as follows: Denote by \( T(h) \) the trapezoidal sum

\[ T(h) = h \left[ \frac{1}{2} f(a) + f(a + h) + f(a + 2h) + \cdots + f(b - h) + \frac{1}{2} f(b) \right] \]

for the stepsize \( h = \frac{b - a}{n} \), \( n \) a natural number. Further denote by \( \widehat{T}_{k}(h) \) that rational function in \( h^{k} \)

\[ \widehat{T}_{k}(h) = \frac{c_{0} + c_{1} h^{2} + \cdots + c_{\mu} h^{2\mu}}{d_{0} + d_{1} h^{2} + \cdots + d_{\nu} h^{2\nu}}, \quad \mu = \left\lfloor \frac{k}{2} \right\rfloor, \quad \nu = h - \mu \]

which meets the interpolation requirements

\[ \widehat{T}_{k}(h_{j}) = T(h_{j}), \quad j = i, i + 1, \ldots, i + k \]

where

\[ h_{j} = \frac{b - a}{n_{j}}, \quad j = 0, 1, 2, \ldots \]

and

\[ \Re := \{ n_{0}, n_{1}, n_{2}, \ldots \} \]

is an arbitrary sequence of natural numbers satisfying

\[ \frac{n_{i}}{n_{i+1}} \leq \beta < 1, \quad i = 0, 1, 2, \ldots \]

for some constant \( \beta \).

Then the extrapolated values

\[ \widehat{T}_{k} := \widehat{T}_{k}(0) \]

*Editor's note. In this fascicle, prepublication of algorithms from the Numerical Integration series of the Handbook for Automatic Computation is started. Algorithms are published in ALGOL 60 reference language as approved by the IFIP. Contributions in this series should be styled after the most recently published ones.
are taken as approximations of the integral (1). The values \(T_k\) which may be displayed in a triangular tableau

\[
\begin{align*}
T(h_0) &= T_0^0 \\
T(h_1) &= T_0^1 & & T_1^0 \\
T(h_2) &= T_0^2 & & T_1^1 & & T_2^0 \\
& & & & & & \vdots \\
& & & & & & T_{m}^0 \\
T(h_m) &= T_0^m & & & & & & T_{m-1}^m
\end{align*}
\]

(3)

can be calculated recursively from the first column of (3) by means of the following formulae (see [2])

\[
\begin{align*}
\Delta_m^0 &:= T(h_m) \\
V_m^0 &:= T(h_m)
\end{align*}
\]

and for \(k = 1, \ldots, m\)

\[
\begin{align*}
\Delta_k^m &:= \frac{V_{k-1}^m \delta_{k-1}^m}{(h_{m-k})^2 \Delta_{k-1}^m - V_{k-1}^m}, \\
V_k^m &:= \frac{(h_{m-k})^2 \Delta_{k-1}^m \delta_{k-1}^m}{(h_{m-k})^2 \Delta_{k-1}^m - V_{k-1}^m} \\
T_k^{m-k} &:= \sum_{q=0}^{h} \Delta_q^m,
\end{align*}
\]

(5)

with the abbreviation

\[
\delta_{k-1}^m := V_{k-1}^m - \Delta_{k-1}^m.
\]

If the denominator of (5) happens to vanish, \(\Delta_k^m\) and \(V_k^m\) are set to zero.

These formulae are evaluated successively for \(m = 0, 1, 2, \ldots\). The indexing in (4) and (5) indicates the actual sequence of calculations. Note, that for programming (4) and (5) only one linear array for storing the differences \(A_q^m\), \(q = 0, 1, \ldots, m\), is needed.

It is known (see [6]) that the error \(T_k^i - J\) for the integral \(J\) can be expressed in the form

\[
T_k^i - J = h_k^2 \ldots h_k^{i+k} \sigma_{k+1}(h_i, \ldots, h_{i+k})
\]

with

\[
\sigma_{k+1}(h_i, \ldots, h_{i+k}) = (-1)^k [e_{k+1} + O(h_k^2)]
\]

and

\[
\begin{align*}
e_{2q} &= \frac{H_{q+1}^{(q+1)}}{H_q^{(q)}} , & e_{2q+1} &= \frac{H_{q+1}^{(q+1)}}{H_q^{(q)}} ,
\end{align*}
\]

where \(H_{p}^{(q)}\) denotes the Hankel-determinant

\[
H_{p}^{(q)} := \begin{vmatrix} \tau_p & \tau_{p+1} & \cdots & \tau_{p+q-1} \\ \vdots & \ddots & \vdots & \vdots \\ \tau_{p+q-1} & \cdots & \tau_{p+2q-2} \end{vmatrix}
\]