We solve the problem of semiscalar equivalence of polynomial matrices to the Smith canonical form diag(1, φ(x), ..., φ(x)) from the condition that the polynomial φ(x) has simple roots.

This paper is devoted to the study of the structure of special classes of polynomial matrices in application to reducing them to simplest form using semiscalar similarity transformations. P. S. Kazimirs'kii and V. M. Petrichkovich have proved [3, 5] that each matrix over the ring P[x], where P is an algebraically closed field of characteristic zero, can be reduced in this way to triangular form with the invariant factors on the main diagonal. It is obvious that for some classes of matrices such a reduction is not the final form, that is, further simplifications of the matrix are possible using semiscalar similarity transformations. In this connection we consider the class K of polynomial matrices with Smith canonical form diag(φ1(x), φ2(x), ..., φ2(x)) under the hypothesis that all the roots of the polynomial \( φ(x) = φ_2(x)/φ_1(x) \) are simple, and we study their quasidiagonal reduction by application of semiscalar similarity transformations. The form of the matrix constructed here is canonical in a certain sense, since it is the direct sum of triangular blocks with the invariant factors on the main diagonals, and is determined up to diagonal equivalence of certain numerical matrices. Among papers in this area, besides those already mentioned, one should note the papers [1, 2, 4, 6, 7] of P. S. Kazimirs'kii and his students.

We shall assume that the base field \( \mathbb{C} \) of complex numbers is linearly ordered lexicographically. In this paper we shall use the concept of the value of a polynomial matrix \( G(x) \) on the system of roots of the polynomial \( φ(x) \) (for the definition see [4]) and the notation \( M_{G(x)}(φ) \), and we shall always carry out the substitution of the roots of the polynomial \( φ(x) \) in increasing order. If, for example, \( φ(x) \) has only the simple roots \( β_1, ..., β_i, β_i' < ... < β_i'' \), then

\[
M_{G(x)}(φ) = \begin{bmatrix} G(β_i) \\ \vdots \\ G(β_i') \end{bmatrix}.
\]

We recall [3–5] that the matrices \( A(x) \) and \( B(x) \) are semiscalar equivalent (denoted \( A(x) \sim B(x) \)) if there exist invertible matrices \( C, Q(x) \) over \( \mathbb{C} \) and \( C[x] \) respectively such that \( B(x) = CA(x)Q(x) \).

Rectangular matrices \( A \) and \( B \) are diagonally equivalent [7] if there exist nonsingular diagonal matrices \( D_1, D_2 \) of corresponding orders such that \( B = D_1 A D_2 \) (denoted \( A \sim B \)).

We denote the identity matrix of order \( r \) by \( E_r \).

In what follows we assume that the first invariant factors of these matrices are 1. Thus \( K \) will denote the class of matrices over \( \mathbb{C}[x] \) of order \( n \) with Smith form \( \text{diag}(1, φ(x), ..., φ(x)) \) under the hypothesis that the polynomial \( φ(x) \) has no multiple roots.

**Proposition 1.** The matrix \( A(x) \in K \) is semiscalar equivalent to a matrix of the form

\[
C(x) = \begin{bmatrix} 1 \\ c_1(x) & φ(x) \\ \vdots \\ c_{l-1}(x) & φ(x) \end{bmatrix} \oplus φ(x) E_{n-l}, \tag{1}
\]
where \( I = \text{rank } M_{\alpha}(\varphi) \), \( T \) is the transposition operator, \( c_i(\alpha_j) = \delta_{ij} \) for certain roots \( \alpha_0, \alpha_1, ..., \alpha_{l-1} \) of the polynomial \( \varphi(x) \), \( i, j = 0, 1, ..., l-1 \); \( c_0(x) = 1 - c_1(x) - ... - c_{l-1}(x) \); and \( \delta_{ij} \) is the Kronecker symbol.

The proof is easy to carry out using Theorem 1 of [5] and Propositions 1, 3, and 4 of [4, § 2 p. II].

**Proposition 2.** If, in the matrices

\[
C_m(x) = \begin{bmatrix}
1 \\
c_1^{(m)}(x) \\
\vdots \\
c_{l-1}^{(m)}(x)
\end{bmatrix} \varphi(x)
\]

of class \( K \), for certain roots \( \alpha_0, \alpha_1, ..., \alpha_{l-1} \) of the polynomial \( \varphi(x) \) the conditions

\[
c_i^{(m)}(\alpha_j) = \delta_{ij}, \quad i, j = 0, 1, ..., l-1,
\]

and

\[
c_0^{(m)}(x) = 1 - c_1^{(m)}(x) - ... - c_{l-1}^{(m)}(x)
\]

hold, then the equivalence \( C_1(x) \sim C_2(x) \Leftrightarrow M_{c_1^{(m)}(x)}(\varphi) \stackrel{d}{=} M_{c_2^{(m)}(x)}(\varphi) \) also holds, where

\[
C_1(x) \sim C_2(x)
\]

**Proof.** Let \( C_1(x) \sim C_2(x) \). From the obvious equality

\[
SC_1(x) = C_2(x) R(x),
\]

where \( S \) and \( R(x) \) are invertible matrices, using transposition and passing to the matrices of values on the system of roots of the polynomials \( \varphi(x) \), taking account of Proposition 1 of [4, § 2 p. II], we obtain the equality

\[
\left( \prod_{p=1}^m M_{\alpha_p}^{(1)}(\varphi) \right) S^T = \text{diag}(r_1, ..., r_t) \prod_{p=1}^m M_{\alpha_p}^{(2)}(\varphi)
\]

where \( r_p = r_1(\beta_p) \neq 0; \ r_1(x) \) is element \((1, 1)\) of the matrix \( R(x) \); and \( \beta_p \) are all the roots of the polynomial \( \varphi(x) \), \( p = 1, t \). We can then write

\[
\left( M_{c_1^{(m)}(x)}(\varphi) \right) S' = \text{diag}(r_1, ..., r_t) \prod_{p=1}^m M_{c_2^{(m)}(x)}(\varphi),
\]

where \( S' \) is a diagonal matrix with \( \det S' \neq 0 \), since \( M_{c_1^{(m)}(x)}(\alpha_0, \alpha_1, ..., \alpha_{l-1}) = E_t \), \( m = 1, 2 \). The first part of the proposition is now proved.

Now suppose conversely that \( M_{c_1^{(m)}(x)}(\varphi) \sim M_{c_2^{(m)}(x)}(\varphi) \). It is then easy to verify that the following equality holds:

\[
\left( \prod_{p=1}^m M_{\alpha_p}^{(1)}(\varphi) \right) \| a_{i,j} \|_n = \text{diag}(r_1, ..., r_t) \prod_{p=1}^m M_{\alpha_p}^{(2)}(\varphi)
\]

Here all the numbers \( r_1, ..., r_t \) are nonzero; the matrix \( \| a_{i,j} \|_n \) is nonsingular; \( a_{i+1,i+1} = \ldots = a_{nn} = 1; \ a_{pq} = 0 \) when \( p \neq q \) and \( p \neq 2, l; q \neq 1 \). From the elements of the matrices \( \| a_{i,j} \|_n \), \( C_1(x) \), and \( C_2(x) \) we construct the matrix \( \| r_{i,j}(x) \|_n \) as follows:

\[
r_{i,j}(x) = a_{i,j} + \sum_{k=2}^n a_{i,k} c_{k-1}^{(1)}(x), \quad r_{i,1}(x) = a_{i,1} \varphi(x), \quad r_{u,0}(x) = a_{u,0} - a_{u,1} c_1^{(2)}(x),
\]

\[
r_{u,v}(x) = (a_{u,v} c_{v-1}^{(1)}(x) - r_{1,1}(x) c_{v-1}^{(2)}(x)) / \varphi(x), \quad u, v = 2, n.
\]

It follows from Eq. (4) that \( r_{i,j}(x) \in \mathbb{C}[x], \ v = 2, n, \) that is, \( \| r_{i,j}(x) \|_n \in M_n(\mathbb{C}[x]) \), and we can write

\[
C_1(x) \| a_{i,j} \|_n = \| r_{i,j}(x) \|_n C_2(x),
\]

leading to Eq. (3). The proposition is now proved.

2800