THE AXISYMMETRIC PROBLEM OF DETERMINING THE RESIDUAL STRESSES IN THICK PLATES

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We propose a method of determining the residual (technological) stresses in structural elements that can be regarded as plates in a computational model. The initial data are the equations of mechanics for bodies with initial stresses and experimental information obtained using nondestructive physical testing.

The residual stresses that arise as the result of various technological processes have a significant effect on the strength and reliability of structures. Because of the complexity of the physico-chemical processes that accompany manufacturing, such as welding of structural elements, there is not yet a model to account for these processes in their full complexity and compute the residual strain and stress fields. Nondestructive physical methods, except in simple cases, also do not make it possible to determine the full picture of the distribution of residual stresses. In this connection experimental-computational methods have been proposed in the literature for studying residual stresses in structural elements, a brief analysis of which can be found in [4], in particular, a method based on solving inverse problems of the mechanics of bodies with initial stresses by use of experimental information obtained through nondestructive physical methods (magnetic, ultrasound, polarization-optical methods, and the like) [3]. We use this method below to propose a method of determining the residual (technological) stresses in thick plates.

To obtain the basic relations that describe its stress-strain state, we represent the components of the small strain tensor \( \epsilon_{ij} \) in a Cartesian coordinate system \( x_1, x_2, x_3 \) in the form [2]

\[
\epsilon_{ij} = \epsilon^0_{ij} + \epsilon_0^0,
\]

Here \( \epsilon_{ij} \) are the components of the total strain tensor; \( \epsilon^0_{ij} \) are the components of the conditional plastic strain tensor, whose incompatibility causes the residual stresses; \( \epsilon^0_{ij} \) are the components of the elastic strain tensor connected with the components of the residual stresses \( \sigma_{ij} \) by the relations

\[
\epsilon^0_{ij} = \frac{1}{E} \left[ (1 + \nu) \sigma_{ij} - \nu \delta_{ij} \sigma_{kk} \right],
\]

where \( E \) is the Young's modulus, \( \nu \) is the Poisson coefficient, and \( \delta_{ij} \) is the Kronecker symbol. Taking account of the expressions (1) and (2), we write

\[
\sigma_{ij} = 2\mu(\epsilon_{ij} - \epsilon^0_{ij}) + \lambda \delta_{ij} (\theta - \theta^0),
\]

where \( \theta = \epsilon_{11} + \epsilon_{22} + \epsilon_{33} \); \( \theta^0 = \epsilon^0_{11} + \epsilon^0_{22} + \epsilon^0_{33} \), and \( \lambda, \mu \) are the Lamé coefficients.

Applying the geometric relations that connect the components of the strain tensor \( \epsilon_{ij} \) with the components of the displacement vector \( u_i \), we give the formulas for determining the stresses in the layer in the form

\[
\sigma_{ij} = 2\mu \epsilon_{ij} + \lambda \delta_{ij} \text{div} u - 2\mu \epsilon^0_{ij} - \lambda \delta_{ij} \theta^0.
\]

Substituting the expressions (3) into the equilibrium equations, we obtain a system of resolvent equations in displacements. This system has the following form in cylindrical coordinates \( r, \varphi, z \):

\[
\mu \nabla^2 u_r + (\lambda + \mu) \frac{\partial \theta}{\partial r} - \frac{\mu}{r^2} u_r - 2\mu \frac{\partial u_r}{\partial r} + 2\mu \frac{\partial \epsilon^0_{rr}}{\partial r} - 2\mu \frac{\partial \epsilon^0_{\varphi \varphi}}{\partial \varphi} - \frac{2\mu}{r} \left( \partial \epsilon^0_{rr} + \lambda \theta^0 \right) - \frac{2\mu}{r} \left( \partial \epsilon^0_{\varphi \varphi} + \epsilon^0_{rr} - \epsilon^0_{\varphi \varphi} \right) = 0,
\]


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\[
\mu \nabla^2 u_\phi + \frac{\lambda + \mu}{r} \frac{\partial \theta}{\partial r} - \frac{\mu e_{\phi \phi}}{r^2} + 2\mu \frac{\partial u_r}{\partial r} - 2\mu \left( \frac{\partial e_{\phi \phi}}{\partial r} - \frac{\partial e_{\phi z}}{\partial z} \right) - \frac{1}{r} \left( 2\mu \frac{\partial e_{\phi \phi}}{\partial r} + \lambda \frac{\partial \theta}{\partial r} + 4\mu e_{\phi \phi} \right) = 0,
\]

\[
\mu \nabla^2 u_r + (\lambda + \mu) \frac{\partial \theta}{\partial z} - 2\mu \left( \frac{\partial e_{\phi \phi}}{\partial r} + \frac{\partial e_{\phi z}}{\partial z} \right) - \frac{\lambda}{r} \frac{\partial \theta}{\partial z} - 2\mu \left( \frac{\partial e_{\phi \phi}}{\partial r} + e_{r r}^0 \right) = 0,
\]

where \( \nabla^2 \) is the Laplacian.

In the case of the axisymmetric problem these equations assume the form

\[
\mu \nabla^2 u_r + (\lambda + \mu) \frac{\partial \theta}{\partial r} - 2\mu \frac{e_{r r}^0}{r^2} - 2\mu \frac{\partial e_{\phi \phi}}{\partial r} - 2\mu \frac{\partial e_{\phi z}}{\partial z} - \frac{2\mu}{r} (e_{r r}^0 - e_{\phi \phi}^0) = 0,
\]

\[
\mu \nabla^2 u_z + (\lambda + \mu) \frac{\partial \theta}{\partial z} - 2\mu \frac{\partial e_{\phi \phi}}{\partial r} - \frac{2\mu}{r} \frac{e_{r r}^0}{r} e_{r r}^0 = 0.
\]

Equations (4) with suitable boundary conditions make it possible to find the connection between the applied strain \( e_{ij}^0 \) and the stresses \( \sigma_{ij} \).

If the technological strain field \( e_{ij}^0 \) can be described by a spherical tensor, then \( e_{rr}^0 = e_{\phi \phi}^0 = e_{z z}^0 = e^0 \), \( e_{r \theta}^0 = e_{r z}^0 = 0 \) and we write the system of equations (4) as:

\[
\mu \nabla^2 u_r + (\lambda + \mu) \frac{\partial \theta}{\partial r} - 2\mu \frac{e_{r r}^0}{r^2} - 2\mu \frac{\partial e_{\phi \phi}}{\partial r} = 0,
\]

\[
\mu \nabla^2 u_z + (\lambda + \mu) \frac{\partial \theta}{\partial z} - 2\mu \frac{\partial e_{\phi \phi}}{\partial r} - \frac{2\mu}{r} \frac{e_{r r}^0}{r} e_{r r}^0 = 0.
\]

or, in vector form, as

\[
\mu \nabla^2 u + (\lambda + \mu) \text{ grad div } u - (3\lambda + 2\mu) \text{ grad } e^0 = 0.
\]

We represent a particular solution of this equation in the form

\[
u = \text{ grad } \phi,
\]

where the scalar-valued function \( \phi \) satisfies Poisson's equation

\[
\nabla^2 \phi = m e^0, \quad m = (1 + \nu)/(1 - \nu).
\]

In general the conditional plastic strain field \( e_{ij}^0 \) can be described as a tensor quantity. If we assume that this tensor is symmetric and that the sum of the normal components \( \theta^0 \) equals zero, then in the case of the axisymmetric problem we write

\[
e_{rr}^0 = e_{\phi \phi}^0 (r, z), \quad e_{\phi \phi}^0 = e_{\phi \phi}^0 (r, z), \quad e_{z z}^0 = -(e_{rr}^0 + e_{\phi \phi}^0), \quad e_{r \theta}^0 = e_{r z}^0 = e_{\phi z}^0 = 0.
\]

In this case the system of equations (4) assumes the form

\[
\mu \nabla^2 u_r + (\lambda + \mu) \frac{\partial \theta}{\partial r} - 2\mu \frac{\partial e_{\phi \phi}^0}{\partial r} + 1 \frac{\partial e_{r r}^0}{\partial r} - \frac{e_{\phi \phi}^0}{r} (e_{r r}^0 - e_{\phi \phi}^0) = 0,
\]

\[
\mu \nabla^2 u_z + (\lambda + \mu) \frac{\partial \theta}{\partial z} + 2\mu \frac{\partial e_{\phi \phi}^0}{\partial z} (e_{r r}^0 + e_{\phi \phi}^0) = 0.
\]

If the components of the tensor \( e_{ij}^0 \) can be assumed independent of the \( z \)-coordinate, then we can write the system of equations (7) as follows:

\[
\mu \nabla^2 u + (\lambda + \mu) \text{ grad div } u - 2\mu \text{ grad } \omega^0 = 0,
\]

where

\[
\omega^0 = e_{rr}^0 (r) + \frac{1}{r} \left[ e_{r z}^0 (r) - e_{\phi \phi}^0 (r) \right] dr.
\]

A particular solution of Eq. (8) can be represented as \( u = \text{ grad } \psi \), where the scalar-valued function \( \psi \) satisfies the equation

2898