Finite-Time Stabilization of Uncertain Nonlinear Planar Systems

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Abstract. A stabilizer is described that renders 0 ∈ ℝ² a global finite-time attractor for a class of nonlinear uncertain two-dimensional dynamical systems. The stabilizer is of discontinuous feedback form: the analytic framework is that of differential inclusions. An adaptive version of the stabilizer is also developed that, for a larger class of systems, guarantees global asymptotic attractivity of the origin.

1. Introduction

As motivation, consider the controlled Duffing-type system with extraneous periodic forcing and unknown real parameters α (≠ 0), a₁, β, ω:

\[ \alpha \ddot{x}(t) + a_1 \dot{x}(t) + a_2 x(t) + a_3 x^3(t) + a_4 \sin \omega t + \beta u(t) = 0. \]  

(1)

In the absence of control (u(·) = 0), this system exhibits highly irregular dynamic behavior, with attracting sets of fractal structure (in time-phase space), for ranges of parameter values (Guckenheimer and Holmes, 1983; Thompson and Stewart, 1986). If the unknown vector of parameters \( a = (a_1, a_2, a_3, a_4) \in \mathbb{R}^4 \) lies in a known compact set \( A \), then solutions of the nonautonomous uncertain system (1) are, a fortiori, solutions of the autonomous differential inclusion

\[ \alpha \ddot{x}(t) + \beta u(t) \in G(x(t), \dot{x}(t)), \]  

(2)

where the set-valued map \( G \) is given by

\[ G : (x, y) \mapsto (1 + |y| + |x| + |x|^3)\tilde{a}B, \]

with \( \tilde{a} := \max\{|a| : a \in A\} \). Here \( |a| := \max\{|a_1|, |a_2|, |a_3|, |a_4|\} \), and \( \tilde{B} \) denotes the closure of the open unit ball \( B \) centered on the origin in \( \mathbb{R}^2 \).

Clearly, system (1) is but one particular example of a nonautonomous uncertain system that can be subsumed by a differential inclusion of form (2). More generally, any scalar two-dimensional nonautonomous uncertain system of the form
\[
\alpha \dot{x}(t) + g(t, x(t), \dot{x}(t)) + \beta u(t) = 0
\]

can be embedded in an autonomous inclusion of form (2) with known right-hand side, provided that the unknown function \( g \) (assumed measurable in its first argument) is bounded by a known function \( \gamma \) in the sense that, for almost all \( t \in \mathbb{R} \),

\[
|g(t, x, y)| \leq \gamma(x, y) \quad \forall (x, y) \in \mathbb{R}^2.
\]

In such cases, we may simply define \( g \) as the map \( (x, y) \mapsto \gamma(x, y)\hat{B} \).

Our contribution is to the study of stabilizability properties for systems of form (2), with the underlying assumption that the set-valued map \( g \) is known, is continuous on \( \mathbb{R}^2 \), and takes nonempty, convex, and compact values (compact intervals of \( \mathbb{R} \)).

We first consider, in section 3, the case in which \( \alpha ;\neq 0 \) is known (we may assume \( \alpha = 1 \) without loss of generality) and \( \beta \) is bounded from below by a known constant \( b > 0 \). Under these conditions, we demonstrate that the inclusion system is finite-time stabilizable: in particular, a discontinuous state feedback strategy is constructed which guarantees that, for every initial-data pair \((x(0), \dot{x}(0)) = (x^0, y^0) \in \mathbb{R}^2\), every state solution \((x(t), \dot{x}(t))\) of system (2) has a maximal interval of existence \([0, \infty)\) and there exists a (calculable) scalar \( T(x^0, y^0) \geq 0 \) such that \((x(t), \dot{x}(t)) = (0, 0)\) for all \( t \geq T(x^0, y^0) \).

In section 4, we weaken the hypotheses on \( \alpha \) and \( \beta \). There, we assume only that both parameters are nonzero. In this case, an adaptive modification to the strategy of section 3 is proposed that preserves the zero state as a global attractor. However, we are only able to establish that the nature of the attractor is asymptotic: whether or not it is finite-time or, indeed, even exponentially attractive is an open question.

2. Formulation

The class of planar systems to be considered is the following:

\[
\begin{align*}
\dot{x}(t) &= y(t), \quad \alpha \dot{y}(t) + \beta u(t) \in \mathcal{G}(x(t), y(t)) \\
(x(0), y(0)) &= (x^0, y^0) \in \mathbb{R}^2.
\end{align*}
\]

We suppose that the full state \((x(t), y(t))\) is available for feedback.

**Assumption 1.** The map \((x, y) \mapsto \mathcal{G}(x, y) \subseteq \mathbb{R}\) is continuous on \( \mathbb{R}^2 \) and takes nonempty, convex, and compact values.

**Remark.** \( \mathcal{G} \) is continuous on \( \mathbb{R}^2 \) if it is both upper and lower semicontinuous at every \((\bar{x}, \bar{y}) \in \mathbb{R}^2\): \( \mathcal{G} \) is upper semicontinuous at \((\bar{x}, \bar{y})\) if, for each \( \epsilon > 0 \), there exists \( \delta > 0 \) such that \( \mathcal{G}(x, y) \subseteq \mathcal{G}(\bar{x}, \bar{y}) + \epsilon B \) for all \((x, y) \in (\bar{x}, \bar{y}) + \delta B\); \( \mathcal{G} \) is lower semicontinuous at \((\bar{x}, \bar{y})\) if, for every sequence \( \{(x_n, y_n)\} \) converging to \((\bar{x}, \bar{y})\) and for every \( \phi \in \mathcal{G}(\bar{x}, \bar{y}) \), there exists a sequence \( \{\phi_n \in \mathcal{G}(x_n, y_n)\} \) converging to \( \phi \) (see Aubin and Cellina, 1984, for further details).

The primary objective is to determine a feedback strategy that renders the zero state of system (3) globally attractive.