ON THE CONVERGENCE OF THE EVEN PART OF THE BRANCHED CONTINUED FRACTION EXPANSION OF A RATIO OF LAURICELLA HYPERGEOMETRIC FUNCTIONS $F_D^{(N)}$

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We study the uniform and absolute convergence of the even part of the branched continued fraction that represents a ratio of Lauricella hypergeometric functions.

Lauricella [3] defined four hypergeometric functions of several variables using multiple power series. The purpose of the present paper is to study the Lauricella function

$$F(DN)(a, b_1, \ldots, b_N; c; z_1, \ldots, z_N) = \sum_{k_1, \ldots, k_N = 0}^{\infty} \frac{(a)_{k_1} \cdots (b_1)_{k_1} \cdots (b_N)_{k_N}}{(c)_{k_1 + \cdots + k_N} k_1! \cdots k_N!} z_1^{k_1} \cdots z_N^{k_N}$$

where $a, b_1, \ldots, b_N, c$ are complex constants with $c \neq 0, -1, -2, \ldots$; $z_1, \ldots, z_N$ are complex variables; $(\alpha)_k = \alpha(\alpha + 1) \cdots (\alpha + k - 1)$ is the Pochhammer symbol; and $(\alpha)_0 = 1$. The series (1) converges in the polydisk $\{(z_1, \ldots, z_N) \in \mathbb{C}^N : |z_i| < 1, i = 1, N\}$.

Molnar [2] has constructed the expansion of the ratio of Lauricella hypergeometric functions

$$\frac{F(DN)(a + 1, b_1 + 1, b_2, \ldots, b_N; c + 1; z_1, \ldots, z_N)}{F(DN)(a, b_1, \ldots, b_N; c; z_1, \ldots, z_N)}$$

in a branched continued fraction.

The even part of the branched continued fraction with the sequence of approximants $\{f_k\}$ is a branched continued fraction for whose sequence of approximants $\{g_k\}$ the equality $g_k = f_{2k}$ holds, $k = 1, 2, \ldots$. The even part of the branched continued fraction into which the ratio of the hypergeometric functions (2) is expanded is a branched continued fraction of the form

$$d_0(z) + a_0(b_0(z) + \prod_{k=1}^{N} \sum_{i_k=1}^{N} \frac{a_{i(k)}(z)}{b_{i(k)}(z)})^{-1}$$

where

$$d_0(z) = \frac{a}{c} (1 - z_1), \quad a_0 = 1 - \frac{a}{c}, \quad b_0(z) = 1 + \sum_{i_1=1}^{N} \frac{b_{i_1} + p_{i(1)}}{a(1 - z_{i_1})} z_{i_1}$$

$$a_{i(n)}(z) = \frac{(c - a + n)(b_{i(n)} + p_{i(n)})z_{i(n)}}{a^2(1 - z_{i(n)})^2}$$

$$b_{i(n)}(z) = 1 + \frac{c - a + n}{a(1 - z_{i(n)})} + \sum_{i_{n+1}=1}^{N} \frac{(b_{i_{n+1}} + p_{i(n+1)})z_{i_{n+1}}}{a(1 - z_{i_{n+1}})}$$

$i(n) = i_1i_2\ldots i_n$ is a multi-index $i_k \in \overline{1, N}$, $k = 1, n, n = 1, \infty$, $p_{i(n)} = \alpha_{i(n)} + \delta_{i(n)}^1$, $\alpha_{i(1)} = 0$, $\alpha_{i(n)}$ is the number of indices $i_n$ in the multi-index $i(n - 1)$, if $n \geq 2$, and $\delta_{i(n)}^1$ is the Kronecker delta.

Let us consider the convergence of the branched continued fraction (3).

We obtain an upper bound for the quantity $|a|^2 |a_{i(n)}(z)|$ for all admissible sets of indices:

$$|a|^2 |a_{i(n)}(z)| = \left| \frac{c - a + n}{1 - |z_{i(n)}|^2} \right| |z_{i(n)}| \leq \left( 1 + |c - a| \right) \left( 1 + |b_{i(n)}| \right) \frac{n^2 |z_{i(n)}|}{|1 - |z_{i(n)}|^2|}$$


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Consider the hyperplane
\[ (c - a + n)v_i + \sum_{j=1}^{N} (b_j + p_j)u_j + a - \sum_{j=1}^{N} b_j - n = 0, \quad i = 1, N, \quad v \in \mathbb{C}^N, \] (4)
where \( n \) is an arbitrary positive integer and \( p_1, p_2, \ldots, p_n \) are nonnegative integers such that \( 0 \leq p_i \leq n, \quad i = 1, N, \)
\[ \sum_{j=1}^{N} p_j = n. \]

Let \( d_i, \quad i = \overline{1, N} \), be the distances from the origin to the hyperplanes (4):
\[ d_i = \left| a - \sum_{j=1}^{N} b_j - n \right| \left( \sum_{j=1}^{N} |b_j + p_j|^2 + |c - a + n + b_i + p_i|^2 \right)^{-1/2}. \]

We find a lower bound for \( d_i, \quad i = 1, N \):
\[ d_i \geq \frac{|a - \sum_{j=1}^{N} b_j - n|}{\sum_{j \neq i}^{N} |b_j + p_j| + |c - a + n + b_i + p_i|} \geq \frac{|a - \sum_{j=1}^{N} b_j - n|}{\sum_{j=1}^{N} |b_j| + |c - a| + 2n} . \]

Let
\[ r = \sup_{n \in \mathbb{N}} \left\{ \left( \sum_{j=1}^{N} |b_j| + |c - a| + 2n \right) |a - \sum_{j=1}^{N} b_j - n|^{-1} \right\} . \]

Then for an arbitrary positive \( \varepsilon \) the following inequalities hold: \( d_i > \frac{1}{r + \varepsilon}, \quad i = 1, N. \)

Hence, in the domain
\[ D_{\varepsilon} = \left\{ z \in \mathbb{C}^N : |1 - z_i| \geq r + \varepsilon, \quad i = \overline{1, N} \right\} , \]
\( b_{i(n)}(z) \neq 0 \) for all indices and \( b_0(z) \neq 0. \)

Since the function
\[ \hat{b}_{i(n)}(v) := \left| (c - a + n)u_i + \sum_{i_{n+1}=1}^{N} (b_{i_{n+1}} + p_{i(n+1)})u_{i_{n+1}} + a - \sum_{j=1}^{N} b_j - n \right| \]
is continuous on the compact set \( K = \left\{ v \in \mathbb{C}^N : \left| u_i \right| \leq \frac{1}{r + \varepsilon}, \quad i = \overline{1, N} \right\} , \) it attains its smallest value at a certain point \( v^* \in K. \)

Let \( \beta_{i(n)} = \frac{1}{n} \hat{b}_{i(n)}(v^*). \) We shall show that all \( \beta_{i(n)} \) are bounded below by a positive number. Since \( |v_i^* - 1| > \frac{1}{r + \varepsilon} \) and \( |v_i^*| \leq \frac{1}{r + \varepsilon} \), it follows that \( m^*_i := |v_i^* - 1| - \max_{j=1, N} |u_j^*| > 0, \quad i = \overline{1, N}. \) Let \( \delta \) be a positive number such that \( \delta < m^*_i. \) For all sufficiently large \( n \) for which
\[ \delta > \frac{1}{n} |a - \sum_{j=1}^{N} b_j + (c - a)u_i^* + \sum_{j=1}^{N} b_j v_j^*|. \]