SPECTRAL PROPERTIES OF GENERATORS OF THE C_0-GROUPS
OVER TENSOR PRODUCTS OF BANACH SPACES

V. A. Ryazhs'ka and O. V. Lopushans'kii

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We study the properties of common spectral subspaces of the \( N \) generators of one-parameter \((C_0)\)-groups on different Banach spaces. The method of study is based on the functional calculus in the convolution algebra \( L^1(\mathbb{R}^N) \). We establish a theorem on the equality of the common spectra of the generators and their restrictions to common spectral subspaces.

1. Consider a finite set of Banach spaces \( \{ (X_n, \| \cdot \|_{x^n}) \}_{n=1}^N \) over the field \( \mathbb{C} \) of complex numbers. Let \( \otimes_n X^n \equiv X^1 \otimes \ldots \otimes X^N \) be their tensor product with the projective norm \( \| w \|_{\otimes_n X^n} \equiv \inf \sum_{j=1}^J \| x_j^1 \|_{X^1} \cdot \ldots \cdot \| x_j^N \|_{X^N}, \) where inf is taken over all representations of the element \( w \in \otimes_n X^n \) as a finite sum \( w = \sum_{j=1}^J \otimes_n x_j^j \), in which we have used the notation \( \otimes_n x_j^j \equiv x_j^1 \otimes \ldots \otimes x_j^N \). The corresponding completion of the tensor product will be denoted by \( \widetilde{\otimes_n X^n} \equiv \widetilde{X^1} \otimes \ldots \otimes \widetilde{X^N} \).

Suppose that a one-parameter \((C_0)\)-group is defined on each of the spaces \( X^n \):

\[
R \ni t_n \rightarrow e^{-i t_n A_n} \in L(X^n)
\]

with generator \(-i A_n\), where \( L(X^n) \) is the algebra of bounded linear operators on the space \( X^n \). The operators \( A_n : \mathcal{B}(A_n) \subset X^n \rightarrow X^n \), as is known, are closed and have dense domains \( \mathcal{D}(A_n) \).

On the space \( \widetilde{\otimes_n X^n} \) we assign to the operator \( A_n \) an operator of the form \( \mathfrak{d}_n \equiv I_1 \otimes \ldots \otimes A_n \otimes \ldots \otimes I_N \), which is also closed and has dense domain \( \mathcal{D}(\mathfrak{d}_n) \). The common domain of definition \( \bigcap_{n=1}^N \mathcal{D}(\mathfrak{d}_n) \) of the set \( \mathfrak{d} \equiv [\mathfrak{d}_1, \ldots, \mathfrak{d}_N] \) is dense in the space \( \widetilde{\otimes_n X^n} \) and on it the operators \( \mathfrak{d}_n \) commute with one another [3].

To the operator \( \mathfrak{d}_n \) one can assign the one-parameter \((C_0)\)-group \( e^{-i t_n \mathfrak{d}_n} \equiv I_1 \otimes \ldots \otimes e^{-i t_n A_n} \otimes \ldots \otimes I_N \) with generator \(-i \mathfrak{d}_n \) in the algebra \( L(\widetilde{\otimes_n X^n}) \) of bounded linear operators on the space \( \widetilde{\otimes_n X^n} \). The set of groups \( \{ e^{-i t_n \mathfrak{d}_n} \}_{t_n=1}^N \) commutes over \( \widetilde{\otimes_n X^n} \), so that their product can be written as

\[
e^{-i t \mathfrak{d}} \equiv e^{-i t_1 A_1} \otimes \ldots \otimes e^{-i t_N A_N}, \quad t \cdot \mathfrak{d} \equiv t_1 \mathfrak{d}_1 + \ldots + t_N \mathfrak{d}_N.
\]

Of course, \( R^N \ni t \rightarrow e^{-i t \mathfrak{d}}, \) where \( t \equiv (t_1, \ldots, t_N), \) is an \( N \)-parameter \((C_0)\)-group over the space \( \widetilde{\otimes_n X^n} \).

In the present paper we use the functional calculus developed for the set of operators \( \mathfrak{d} \) in the Fourier-image of the convolution algebra \( L^1(\mathbb{R}^N) \) of integrable functions \( f(t) = f(t_1, \ldots, t_N) \) over the space \( \mathbb{R}^N \) to study the properties of their common spectral subspaces. We assume that each of the groups \( e^{-i t_n A_n} \) is uniformly bounded with respect to \( t_n \in R \) in the algebra \( L(X^n) \).

2. In the complex Banach space \( L^1(\mathbb{R}^N) \) of integrable functions \( f(t) = f(t_1, \ldots, t_N) \) on \( \mathbb{R}^N \) with the norm

\[
\| f \|_{L^1(\mathbb{R}^N)} \equiv \int_{\mathbb{R}^N} |f(t)| \, dt,
\]

we assign to any number \( \nu > 0 \) the subspace of functions

\[
\exp^\nu(D) \equiv \left\{ f \in L^1(\mathbb{R}^N) : \| f \|_{L^\nu} < \infty \right\} \text{ with norm } \| f \|_{L^\nu} \equiv \sum_{|k|=0}^\infty \frac{1}{\nu^{|k|}} \| D^k f \|_{L^1(\mathbb{R}^N)}. \]


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where \( k = (k_1, \ldots, k_N) \in \mathbb{Z}_+^N, |k| = k_1 + \ldots + k_N, D^k \equiv D_1^{k_1} \cdots D_N^{k_N} \) and \( D_n = \frac{\partial}{\partial t_n} \). Further let

\[
\exp(D) \equiv \bigcup_{\nu > 0} \exp^\nu(D) = \lim \inf_{\nu \to +\infty} \exp^\nu(D)
\]

be the inductive limit with respect to the inclusion \( \exp^\nu(D) \subset \exp^{\nu+1}(D) \).

Similarly, in the complex space \( L^1(R) \) of integrable functions of one variable \( R \ni t_n \to f(t_n) \) with norm \( \| f \| \) and \( D_n = \frac{\partial}{\partial t_n} \), we distinguish the subspaces \( \exp^\nu(D_n) \) and \( \exp(D_n) \). Since the differentiation operator \( D_n \) on \( L^1(R) \) can be regarded as closed, the spaces \( \exp^\nu(D_n) \) are Banach spaces \([2]\). The following structural proposition holds \([2]\).

**Lemma 1.** The following topological isomorphisms hold:

\[
\exp^\nu(D) \simeq \exp^\nu(D_1) \otimes \cdots \otimes \exp^\nu(D_N), \quad \exp(D) \simeq \exp(D_1) \otimes \cdots \otimes \exp(D_N),
\]

and the first of them is an isometry.

**Proof.** The isometric isomorphism \( L^1(R^N) \cong L^1(R) \otimes \cdots \otimes L^1(R) \) holds. Indeed, by Fubini's theorem, \( L^1(R^N) \) coincides with \( L^1(R; L^1(R^{N-1})) \), the space of \( L^1(R^{N-1}) \)-valued functions \( R \ni t_n \to f(t_1, \ldots, t_{N-1}, t_n) \in L^1(R^{N-1}) \) with norm

\[
\left\| f(t_1, \ldots, t_{N-1}, t_n) \right\|_{L^1(R^{N-1})} = \int_{R^{N-1}} |f(t_1, \ldots, t_{N-1}, t_n)| dt_1 \ldots dt_{N-1}.
\]

This norm equals the norm of the projective tensor product \( L^1(R) \otimes L^1(R^{N-1}) \), that is, the isometric isomorphism \( L^1(R; L^1(R^{N-1})) \cong L^1(R) \otimes L^1(R^{N-1}) \) holds \([7]\). Repeating such reasoning, we arrive at the required isometry.

The isometric isomorphism (1) now becomes a corollary of the following equalities

\[
\| f \|_v = \sum_{|k| = 0}^{\infty} \frac{1}{\nu^{|k|}} \| D^k f \|_{L^1(R^N)} = \sum_{|k| = 0}^{\infty} \frac{1}{\nu^{|k|}} \inf \left[ \sum_j \| f_{1,j}(t_1) \|_{L^1(R)} \cdots \| f_{N,j}(t_N) \|_{L^1(R)} \right] = \inf \sum_j \| f_{1,j} \|_v \cdots \| f_{N,j} \|_v = \| f \| \otimes \exp^\nu(D_n),
\]

where the infimum is taken over all representations of \( f \in \otimes_n \exp^\nu(D_n) \) as \( f(t) = \sum_j f_{1,j}(t_1) \cdots f_{N,j}(t_N) \), a finite sum of products of functions of a single variable \( f_{n,j}(t_n) \in \exp^\nu(D_n) \).

The second topological isomorphism follows from Lemma 7 of \([3]\).

**Theorem 1.** (a) The spaces \( \exp^\nu(D) \) are Banach spaces and coincide with the restriction to the real space \( R^N \) of the class of entire holomorphic functions \( f(t + is) = f(t_1 + is_1, \ldots, t_N + is_N) \) of exponential type \( < v \) that are uniformly bounded on \( R^N \).

(b) The spaces \( \exp^\nu(D) \) are ideals in the convolution algebra \( L^1(R^N) \) and are invariant under the partial derivative operator \( D_n \). Moreover, each of the operators \( D_n \) on the space \( \exp^\nu(D) \) has norm \( < v \).

(c) The ideal \( \exp(D) \) is continuously and densely embedded in the algebra \( L^1(R^N) \).

**Proof.** The completeness of \( \exp^\nu(D) \) follows immediately from the isomorphism (1). The proof of the existence of a holomorphic extension of the function of one variable \( f(t_n) \) from the space \( \exp^\nu(D_n) \) to entire functions of exponential type is based on Sobolev's theorem and Bernstein's inequality and is known \([6]\). Therefore the functions \( f(t) \) can also be extended from the tensor product \( \otimes_n \exp^\nu(D_n) \) to entire functions \( f(t + is) \) of exponential type. By this inequality, if the type of the analytically extended function \( f(t + is) \) is \( \mu \), then

\[
\| D^j \|_{L^1(R^N)} \leq \mu^j \| f \|_{L^1(R^N)}
\]

for all \( j \in \mathbb{Z}_+ \). Therefore \( f(t) \in \exp^\nu(D) \) when \( \mu < v \). Conversely, for any function \( f(t) \in \exp^\nu(D) \) we have

\[
\| D_n f \|_v = \nu \sum_{|k| = 1}^{\infty} \frac{1}{\nu^{|k|}} \| D^k f \|_{L^1(R^N)} < \nu \| f \|_v.
\]