ON THE SET OF POSITIVE DEFLECTIONS OF FUNCTIONS
MEROMORPHIC IN THE UNIT DISK

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For an arbitrary \( \lambda \in (0, 1/2) \) we construct a function \( f(z) \) of lower order \( \lambda[f] = \lambda \) that is meromorphic in the disk \( D = \{ z : |z| < 1 \} \) and such that the set \( \Omega(f) \) of positive deflections of the function \( f \) (in the sense of V. P. Petrenko) has positive logarithmic capacity.

We shall adhere to the standard notation of Nevanlinna theory [3]. For any function \( f(z) \) that is meromorphic in the disk \( D = \{ z : |z| < 1 \} \) we set

\[
L(r, a, f) = \begin{cases} 
\max_{|z|=r} \frac{1}{|f(z) - a|}, & a \neq \infty, \\
\max_{|z|=r} |f(z)|, & a = \infty.
\end{cases}
\]

In a definition of Petrenko, the deflection of the function \( f(z) \) with respect to the point \( a \) is the quantity \( \beta(a, f) = \liminf_{r \to 1} \frac{L(r, a, f)}{T(r, f)} \). The set \( \Omega(f) = \{ a : \beta(a, f) > 0 \} \) is the set of positive deflections of \( f(z) \). The order \( \rho[f] \) and the lower order \( \lambda[f] \) of the function \( f \) are defined as

\[
\rho[f] = \limsup_{r \to 1} \frac{\ln T(r, f)}{-\ln(1 - r)}, \quad \lambda[f] = \liminf_{r \to 1} \frac{\ln T(r, f)}{-\ln(1 - r)}.
\]

Petrenko ([3], p. 93) has shown that for any \( \lambda \in [0, \infty) \) there exists a function \( f_\lambda(z) \) meromorphic in \( D \) of lower order \( \lambda \) for which the set \( \Omega(f_\lambda) \) has cardinality of the continuum. He also proved ([3], p. 98) that when \( \lambda[f] > 6 \), the set \( \Omega(f) \) has logarithmic capacity zero. On the other hand, for the function \( g(z) = \exp \frac{1 + z}{1 - z} \), we have \( A = \left\{ e^{\theta} : \theta = \cot \frac{\varphi}{2}, \ 0 < \varphi < 2\pi \right\} \subset \Omega(g) \). That is, \( \Omega(g) \) has positive logarithmic capacity, and \( \max T(r, f) < +\infty \).

Petrenko has posed the problem of finding a sharp limit on the growth of meromorphic functions in \( D \) for which the set of positive deflections has logarithmic capacity zero. In 1924 Szegő proved ([1], p. 302) that the logarithmic capacity of any compact set equals its transfinite diameter. Fekete gives the following method of computing the transfinite diameter of a compact set \( E \) ([1], p. 285).

Let \( V(z_1, z_2, \ldots, z_n) = \prod_{1 \leq k < l \leq n} (z_k - z_l), \ n \geq 2, \ z_i \in E, \ i = 1, n, \ V_n = V_n(E) = \max_{z_i \in E} |V(z_1, \ldots, z_n)|, \ d_n = V_n^{2/(n-1)}. \) Then \( d_n \) is a nonincreasing sequence, so that the limit \( \lim_{n \to \infty} d_n = d \) exists. Then \( d \) is the value of the transfinite diameter.

In what follows, brackets in a formula denote the integer part of the expression they contain, and \( C_m^n = \frac{m!}{n!(m-n)!} \) is the binomial coefficient.

Let \( E \) be the set of numbers of the form

\[
x = 1 + \sum_{k=1}^{+\infty} \gamma_k 2^{-[p^k]}, \quad p \in (1, 2), \quad \gamma_k \in \{0, 1\}.
\]

Lemma. The set \( E \) has positive logarithmic capacity.

Proof. It is easy to see that \( E \) is compact. It is obvious that different \( x = x(\gamma_k) \) in \( E \) correspond to different \( (\gamma_k) \).
Consider the set $M_k$ of numbers $x_n$ in $E$ for which $y_m(x_n) = 0$ when $m > k$. It is obvious that $n = 1, 2^k$. The total number of unordered pairs $(x_n, x_j)$ with $n \neq j$ is $C_k^{2^k} = 2^{k-1}(2^k - 1)$. We partition the set of these pairs into $k$ subsets $G_n$ as follows. The set $G_n$ contains the pairs of numbers that differ in exactly $n$ binary digits. The number of pairs in each such subset is $C_k^{2^k-1}$. Indeed a choice of $n$ digits in which the numbers of $G_n$ are to differ can be made in $C_k^{2^k-1}$ ways. Since each $x \in M$ belongs to some pair in $G_n$, one number of the pair can be chosen in $2^{k}$ ways. This number immediately determines its pair in $G_n$. Since the pairs are unordered, we obtain $|G_n| = C_k^{2^k-1}$.

We now give a lower bound on the quantity $\prod_{x_i, x_m \in G_n, i \neq m} |x_i - x_m|$. Fix $j \in \{1, 2, \ldots, k - n + 1\}$. The number of pairs of numbers in $G_n$ such that the first binary digit in which the numbers in the pair differ is the $[p/j]$th is $2^{k-1}C_k^{n-1}$. Again, we can allocate the $n - 1$ remaining digits among the $k - 1$ vacant places in $C_k^{n-1}$ ways. Next, choosing one number of the pair in $2^{k}$ different ways, we count each pair twice.

If the first binary digit in which $x_i$ and $x_m$ differ is the $[p/j]$th, then $|x_m - x_i| \geq 2^{-[p/j]-1}$. Thus,

$$\prod_{x_i, x_m \in G_n, i \neq m} |x_i - x_m| \geq \prod_{j=1}^{k-n+1} 2^{-2^{k-1}C_k^{n-1}([p/j]+1)}.$$

Applying the last estimate, we obtain

$$|V(x_1, \ldots, x_{2^k})| \geq \prod_{n=1}^{k} \prod_{j=1}^{k-n+1} 2^{-2^{k-1}C_k^{n-1}([p/j]+1)} = \prod_{j=1}^{k} \left( \prod_{n=1}^{k-j} 2^{-2^{k-1}C_k^{n-1}([p/j]+1)} \right)^{2^k-1} = \prod_{j=1}^{k} \left( 2^{-2^{k-1}j} \right)^{2^k-1} \geq \prod_{j=1}^{k} 2^{-2^k-j} \cdot p_j = 2^{-2^k} \sum_{j=1}^{k} p_j.$$

Hence

$$d_{2^k} \geq \left( 2^{-C(p)2^k} \right) \frac{1}{2^k} = 2^{-C(p)/1-2^k}.$$

The assertion of the lemma now follows.

**Theorem.** For an arbitrary $\lambda \in [0, 1/2)$ there exists a function $f(z)$ meromorphic in $|z| < 1$ such that $\lambda[f] = \rho[f] = \lambda$ and the capacity of the set of Petrenko positive deflections, $\Omega(f) = \{a : \beta(a, f) > 0\}$, is positive.

**Proof.** In the case $\lambda = 0$ the theorem is trivial: $f(z) = \exp \frac{1+z}{1-z}$. Let $\lambda > 0$. Let us consider a function of the form considered by Petrenko ([3], p. 93). We set $d(z) = \frac{1+z}{1-z}$. For $\theta \in (0, \pi/2)$ we use the notation $\Gamma(\theta) = \{z : |z| < 1, |z + i \cot \theta| = \csc \theta \}$. When $r \geq \tan \frac{\theta}{2}$, $r \in (0, 1)$ let $z_r = z_r(\theta) = \Gamma(\theta) \cap \{z : |z| = r\} \cap \{z : \Re z \geq 0\}$. We assume that $z_r = x_r + iy_r$. From the definition of $\Gamma(\theta)$ we find:

$$|x_r + iy_r + \cot \theta| = \csc \theta \leftrightarrow r^2 + 2y_r \cot \theta = 1.$$ 

Hence $y_r = \frac{1}{2}(1 - r^2) \tan \theta$, $x_r = \sqrt{1 - 2y_r \cot \theta - y_r^2} = \sqrt{r^2 - ((1 - r^2) \tan^2 \theta)/4}$, $r^2 - x_r^2 = \frac{1}{4}(1 - r^2)^2 \tan^2 \theta$.

For $z_r = re^{i\phi}$ we find a lower bound for $|d(z_r)|$.

$$|d(z_r(\theta))| = \frac{|1 - r^2 + 2ir \sin \phi|}{1 + r^2 - 2r \cos \phi} = \frac{|1 - r^2 + i(1 - r^2) \tan \theta|}{(1 - r)^2 + 2(r - x_r)} = \frac{|1 + i \tan \theta|}{1 + \frac{r^2}{(1 + r^2) \tan^2 \theta}}$$

$$\geq \frac{1 + r}{1 - r} \frac{|1 + i \tan \theta|}{(1 + r^2) \tan^2 \theta} \geq \frac{1}{1 + 2 \tan^2 \theta} \frac{K_1(\theta)}{1 - r}.$$

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