ON THE NORMAL STRUCTURE OF THE GENERAL LINEAR GROUP OVER A RING

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The present paper is devoted to the study of normal subgroups of the general linear group over a ring and the centrality of the extension \( \text{St}(n, R) \rightarrow E(n, R) \). The notions of the standard commutator formula and the standard normal structure of \( \text{GL}(n, R) \), \( E(n, R) \), and \( \text{St}(n, R) \) and their relationships are discussed. In particular, it is shown that the normality of \( E(n, R) \) in \( \text{GL}(n, R) \) and the standard distribution of subgroups normalized by \( E(n, R) \) follow from some conditions of linear dependence in \( R \). Also, it is proved that the standardness of the normal structure of \( \text{GL}(n, R) \) and the centrality of \( K_2(n, R) \) in \( \text{St}(n, R) \) follow from the same conditions over a quotient ring \( R/I \), provided that \( \text{st} I \leq n-1 \). Under certain additional assumptions (for example, \( I \) is contained in the Jacobson radical of \( R \)), the converse is also true. The standard technique due to H. Bass, Z. I. Borevich, N. A. Vavilov, L. N. Vaserstein, W. van der Kallen, A. A. Suslin, M. S. Tulenbaev, and others is used and developed in this paper. Bibliography: 21 titles.

INTRODUCTION

The study of the normal structure of the general linear group over general rings was initiated by H. Bass (see [12, 13]) in the middle 60s. In particular, he gave a complete description of the normal subgroups of general linear groups over an arbitrary ring. Also, Bass introduced a new notion of the dimension of rings, the so-called stable rank, and discovered that the main structure results remain valid for groups whose degrees are greater than the stable rank of the ground ring. J. Milnor initiated the study of the Steinberg group in this context (see [6]). A very important contribution to this area of study is due to A. Bak, R. K. Dennis, W. van der Kallen, A. A. Suslin, and L. N. Vaserstein.

The next major breakthrough was triggered by the works of J. S. Wilson [21], I. Z. Golubchik [5], and A. A. Suslin [9]; they discovered that for commutative rings, the above results remain valid starting with degree 3 independently of the dimension of the ground ring. Further results in this direction were obtained by Z. I. Borevich, I. Z. Golubchik, V. I. Kopeiko, A. V. Mikhalev, A. A. Suslin, G. Taddei, L. N. Vaserstein, N. A. Vavilov, and many others. W. van der Kallen and M. S. Tulenbaev proved similar results for the Steinberg group (starting with dimension 4). Also, it should be mentioned that V. N. Gerasimov [4] constructed (in any dimension) an example of a ring such that the elementary subgroup is not normal in the general linear group (and, therefore, the normal structure of the general linear group over such a ring is nonstandard). A complete review of these subjects is given in the book by A. J. Hahn and O. T. O'Meara [16] and in the survey by N. A. Vavilov [20].

At present, the most intriguing problem in the theory of linear groups over rings is to determine the exact class of rings where the behavior of the general linear group and of the Steinberg group is standard, i.e., is almost the same as in the stable situation described by Bass and Milnor. Partial results in this direction are obtained in the present article.

The layout of the article is as follows. In the first section, the main notation and the notions of the standardness of the normal structure of the general linear group and the stable rank of a ring are introduced and elementary properties of them are given. The subsequent sections are devoted to the following:

- the normality of the elementary subgroup in the general group;
- the standard commutator formula;
- the standard distribution of normal subgroups in the general linear group;
- the centrality of the extension \( \text{St}(n, R) \rightarrow E(n, R) \).

An important idea of the article is that the standardness of the normal structure of \( \text{GL}(n, R) \) can be "lifted" modulo an ideal \( I \) if its stable rank is less than \( n \). In particular, we show that the standardness of the normal structure of \( \text{GL}(n, R) \) does not depend on the Jacobson radical \( \text{Rad} R \) of the ring \( R \).
1. THE MAIN NOTATION

1.1. Let $G$ be a group. For elements $z, y \in G$, we denote by $[z, y] = z^{-1}y^{-1}zy$ the commutator of them and by $z^y = y^{-1}zy$ the conjugate of $x$ by $y$. The double commutator $[[z, y], z]$ is denoted by $[z, y, z]$. We write $H \leq G$ to denote that $H$ is a subgroup of $G$ and $H \triangleleft G$ to denote that $H$ is a normal subgroup of $G$. For a subset $X \subseteq G$, we denote by $(X) \leq G$ the subgroup of $G$ generated by $X$ and by $(X)^H$ the smallest subgroup that contains $X$ and is normalized by $H$. For two subgroups $F, H \leq G$, we denote by $[F, H]$ the corresponding relative commutator subgroup generated by all commutators $[f, h]$, $f \in F$, $h \in H$.

1.2. Let $D$ and $G'$ be subgroups of $G$. Recall (see [8]) that a subgroup $F$ of $G'$ is called $D$-perfect if $[F, D] = F$. For a given $D$-perfect subgroup $F$, the largest subgroup $C$ in $G'$ with the property $[C, D] = F$ is called a generic subgroup corresponding to $F$ (see [14]). Clearly, all subgroups from each sandwich $F \leq \ldots \leq C$ are normalized by $D$. We say that $D$ is strongly polynormal in $G$ if for any $H \leq G$ there exists a unique $D$-perfect subgroup $F$ such that $H$ fits into the sandwich $F \leq \ldots \leq C$, where $C$ is a respective generic subgroup.

In the sequel, we often use the following fact easily derived from Theorem 1 in [8].

1.3. Proposition. Let $D$ be a normal perfect subgroup of $G$, i.e., $[D, D] = D \triangleleft G$. Then $D$ is strongly polynormal in $G$ (and hence, in any subgroup of $G$).

1.4. Let $R$ be an associative ring with 1 and let $GL(n, R)$ be the general linear group of degree $n$ over $R$. We always assume that $n \geq 3$. As usual, for a matrix $a \in GL(n, R)$ we denote by $a_{ij}$ the matrix entry at the position $(i, j)$ and by $a^{-1}_{ij}$ the corresponding element of its inverse $a^{-1}$. Quite often we need to use the following notation for the rows and columns of the matrices $a$ and $a^{-1}$: the $i$th rows of these matrices will be denoted by $a_{i*}$ and $a^{-1}_{i*}$, while the $i$th columns will be denoted by $a_{*i}$ and $a^{-1}_{*i}$, respectively. The identity matrix of degree $n$ is denoted by $e = e_n$ and the standard matrix units (i.e., the matrices with 1 at the position $(i, j)$ and zeros at the remaining positions) by $e_{ij}$. The entry of $e$ at the position $(i, j)$ is denoted by $\delta_{ij}$ (the Kronecker symbol).

An elementary transvection $t_{ij}(\xi)$ is a matrix of the form $t_{ij}(\xi) = e + \xi e_{ij}$, $\xi \in R$, $1 \leq i \neq j \leq n$. The subgroup $E(n, R)$ of the general linear group $GL(n, R)$ generated by all elementary transvections $t_{ij}(\xi)$ is called the elementary subgroup. Let $\alpha = (\alpha_1, \ldots, \alpha_n)^T$ be a column and $\beta = (\beta_1, \ldots, \beta_j)$ a row of length $n$ over a ring $R$. For brevity of the notation in proofs, we use the following abridged notation:

$$ t_{ij}(\alpha) = \prod_{j \neq i} t_{ij}(\alpha_j), \quad t_{ij}(\beta) = \prod_{j \neq i} t_{ij}(\beta_j). $$

1.5. Now recall the definitions of certain subgroups of $GL(n, R)$ normalized by $E(n, R)$. For each ideal $I$ of a ring $R$, there is a reduction homomorphism $ho_I : GL(n, R) \to GL(n, R/I)$ such that the image of a matrix $x = (x_{ij})$ is equal to $\bar{x} = (\pi_I(x_{ij}))$. The kernel of the homomorphism $\rho_I$ is denoted by $GL(n, R, I)$ and is called the principal congruence subgroup of level $I$. The inverse image of $C(n, R/I)$ under the reduction homomorphism $\rho_I$ is denoted by $C(n, R, I)$ and is called the full congruence subgroup of level $I$. The relative elementary subgroup of level $I$ is the group

$$ E(n, R, I) = \{t_{ij}(\xi) | \xi \in I, 1 \leq i \neq j \leq n\}^{E(n, R)}. $$

Note that the relative elementary subgroups are (if $n \geq 3$) always $E(n, R)$-perfect. In particular, the group $E(n, R)$ is perfect.

1.6. It turns out that, in the standard situation, the following assertions are true:
1. $E(n, R)$ is a normal subgroup of $GL(n, R)$;
2. for each ideal $I \leq R$, the standard commutator formula

$$ [GL(n, R, I), E(n, R)] = E(n, R, I) $$

Note that the relative elementary subgroups are (if $n \geq 3$) always $E(n, R)$-perfect. In particular, the group $E(n, R)$ is perfect.

1.7. It turns out that, in the standard situation, the following assertions are true:
1. $E(n, R)$ is a normal subgroup of $GL(n, R)$;
2. for each ideal $I \leq R$, the standard commutator formula

$$ [GL(n, R, I), E(n, R)] = E(n, R, I) $$

Note that the relative elementary subgroups are (if $n \geq 3$) always $E(n, R)$-perfect. In particular, the group $E(n, R)$ is perfect.