A Prolegomenon to Mathematical Psychology

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I.

The empirical sciences, characterized by the possibility of systematic observation and experimentation, are classified under two headings: natural and social. The former is supposed to deal with inanimate and nonhuman animate aspects of nature, such as physics, geology, biology, whereas the latter specifically deals with man and all the products of interaction between man and man. Another way of classifying sciences is with reference to the mode of attack: observational and experimental. Objects of the universe which are beyond the capacity of man and his techniques to bring within the bounds of laboratory control can only be "observed"; systematic biology and all the social sciences are instances of this. In particular, social sciences have to presume as a fundamental fact the variability of human nature and will, which challenge the validity of laboratory techniques. Psychology, however, has undertaken a herculean task in the attempt to make an experimental science of itself. Since Wundt's founding of the psychological laboratory in Leipzig in 1879, innumerable laboratories all over the world have been started and diverse aspects of human nature are being studied in this spirit. After several decades of assiduous experimentation, it is interesting to be told by a notable contemporary psychologist, "In a sense it is true to say that through all this vast mangle the very birth-cry of the infant science is still resounding." 1

The reason for this inability to advance must be sought in certain methodological errors. Firstly, psychology has all along been overwhelmed by syntax language and has not been able to develop "object" language. The vocabulary is mostly drawn from speculative philosophy, with frequent eccentricities of subjective interpretation. All the principal categories of present-day psychology are at least as old as Aristotle, and the history of the subject is just a succession of restatements of the same issues. Frequently the dictionary fallacy of circular definition is made.

Secondly, Dantzig's criticism 2 that philosophy lacks the "principle of relativity" holds true for psychology also. The scope of psychology has never been properly defined and the absence of a particular frame of reference has deprived the experimental results of integrity. The confusion that prevails in this subject, although often unrecognized, is evidenced by the multitude of definitions of the subject matter that exist.

Thirdly, the zeal for experimentation has resulted in the mass production of assertions, of which no "positional function" exists; the particular constituents never necessitate generalizations, the instances seldom lead to concepts and the subject lacks "theoretical certainty"; the experimental findings do not bear the stamp of finally, and do not admit of universal application.

Finally, as Bridgman notes, "In the social sciences there is lacking, to such a large extent as to make a difference in the general atmosphere, that disinterested point of view which in the physical sciences we associate with so-called pure science as distinguished from applied science". 3 Practical application is the temptation under which most psychological research is conditioned or hurried through.

In effect, the subject of psychology lacks the logical structure which is essential to all science. One of the essential tasks of scientific endeavour is to determine logical constants which will help systematization and classification of categories and an exact analysis of facts. The spirit of logic or mathematics must be infused into psychology if it is to preserve its integrity.

II.

Experience convinces us that objects and events, and sets of objects and events, often obey logical schemes. This assumption is at the back of all science: experimentation and measurement. The method of representing logical schemes and working out their implications characterizes mathematics; it is a pure hypothetic-deductive system. "We understand the term mathematical science to mean any set of propositions arranged according to a sequence of logical deduction." 4 There must, in the first instance, be a set of axioms or postulates consisting of one or more undefined primitives, and secondly there must be rules of deduction or a system of logic. The axioms need not be obvious truths, as Bertrand Russell points out, but the set of axioms must be consistent, and each axiom must be independent and categorical. As Nwason notes, "axiom written in the acceptable form has the form of a proposition and the characteristics of a mathematical function." It has the function of a "theoretical juice extractor" as Hempel describes it. The actual extractions are done by the formal deductive reasoning, or logical analysis.

What is the province of mathematics? Poincaré wrote: "Mathematics do not study objects but the relation between objects. Matter does not engage their attention. They are interested in form alone." Nothing new, therefore, is discovered or added to the content of our knowledge. It is just a conceptual technique for the scientific understanding of our experiential data. Mathematics is essentially a language: to represent the structures of objects, structures of symbols are built and the "structural properties of symbolic systems" are studied. Thus mathematics helps crystallization of thought and necessitates the fruitful analysis.

III. Groundwork

(1) An attempt is made here to develop psychology on the basis of a small number of sufficient and necessary assumptions or axioms. Before enunciating some axioms, a passing reference must be made to some of the most fruitfully applicable mathematical concepts and conventions. A set is a collection of elements or individuals, wherein a definition obtains determining the "belongingness" or membership character of any particular individual or element with respect to the set. When A and B are

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two sets, and \( x \) any element, we have the following possible relations:

\[ A = B, \text{ i.e., } A \subseteq B \text{ and } B \subseteq A \text{ (identity)}; \]
\[ A \subseteq B, \text{ or } B \supseteq A \text{ (inclusion)}; \]
\[ x \in A \text{ (membership)}; \]
\[ A \cap B, B \Delta A, A + B, x \in A \text{ (negation)}; \]
\[ A \cup B \text{ (product)}; \]
\[ A \cap B \text{ (union)}; \]
\[ A - B \text{ (remainder)}; \]
\[ A = 0 \text{ (null set)}; \]
\[ x \in (a), \text{ i.e., } x = a \text{ (one-element set)}; \]
\[ A \cap B = 0 \text{ (empty intersection)}. \]

All the above concepts have been used in their usual topological sense. Among the axioms that obtain, we note:

If \( x \) is an element and \( S \) any set:

(a) When \( x \in S \), \( U \) is a set of elements containing \( x \) (Hausdorff). Call \( U \) the neighbourhood of \( x \).

(b) If \( x \in U \) and if \( y \) is also an element of \( U \), then \( U \) is the neighbourhood of \( y \) also.

(c) If \( U \) and \( V \) are neighbourhoods such that \( x \in U \cap V \) then there exists a neighbourhood \( W \) such that \( W \subseteq U \cap V \).

(d) If \( x \) and \( y \) are distinct elements, then there exists a neighbourhood of \( x \) not containing \( y \).

The set of elements that proves true to these axioms is termed a topological space. The set theoretical method studies the local properties of this space, such as positional relationships, connectivity, belongingness and qualitative definition. The locally connected continuum is defined by the Jordan curve, which decomposes the plane into precisely two separated regions, one lying inside and the other outside the curve.

The other topological concepts that are employed in this customary significance are such as closed or open sets, bounded sets, complementary sets, derived sets, limit points, interior or exterior points, boundary points, region, cut, connection, ordering.

(2) We thus arrive at the concept of a space or field—"a totality of possibilities of relative positions of practically rigid bodies" (Einstein). But in order to overcome the limitations of treating psychological phenomena as rigid or static, vector viewpoint is necessary.

The psychological space is a dynamic whole comprehending both function and structure, entity and event. The psychological space is a vector field, for the dynamic property of this space originates and determines psychological action: the field forces are the vectors. A vector produces and directs psychological action. It implies a movement from an origin \((A)\) to a terminus or goal \((B)\), thus \(AB = x\). If the movement is negatively valanced, we get \(AB = -x\). Certain assumptions obtain herein:

(a) Every pair of ordered points \( A, B \) whether coincident or not, determines a vector \(AB\); and if \( A \) is any point and \( x \) any vector, there exists only one point \( B \) such that \(AB = x\).

(b) Each pair of vectors \( \alpha, \beta \) determines uniquely a vector \( \gamma \) such that if \( AB = \alpha \), and \( BC = \beta \), then \( AC = \gamma \).

(c) Of two vectors \( \alpha, \beta \), other things being equal, the vector of greater magnitude tends to suppress the vector of lesser magnitude.

(d) Of two vectors \( \alpha, \beta \), other things being equal, the one that easily tends to greatest possible satisfaction, tends to suppress the other that does not.

(3) We can also fruitfully borrow, \textit{mutatis mutandis}, the concept of Fréchet's metric space, where the distance function \( \delta(x, y) \) satisfies the following conditions

\[ \delta(x, y) = 0; \]
\[ \delta(x + y, x) = 0; \]
\[ \delta(x, y) = \delta(y, x); \]
\[ \delta(x, y) + \delta(y, z) > \delta(x, z). \]

Thus by the utilization of the concept of force (vector) and distance, it is possible to evolve the theory of psychological work. Kurt Lewin has indeed shown the way in which topological and vector concepts might be used for psychological research.

IV. Psychological Axiomatics

(1) We will state a set of three axioms, of inclusion, attribution and action respectively.

(i) If \( S \) is any point-set, and \( m \) any member thereof, \( m \) and \( S \) are related by inclusion such that

\[ m = m; \text{ or } m \in (m), \]
\[ m + S; \text{ or } S - m + 0; \]
\[ m \in S. \]

(ii) Whenever \( m \in S \), the relation of inclusion satisfies the following conditions:

\[ m \text{ has the attribute of } m; \text{ i.e., } m \subseteq m; \]
\[ m \text{ has no attribute of } m; \text{ i.e., } m \nsubseteq m; \]
\[ m \text{ has the attribute of } S; \text{ i.e., } m \subseteq S. \]

(iii) Whenever \( m \in S \), it is always an action-system such that

\[ m \text{ acts in accordance with } m \subseteq m; \text{ i.e., } m \subseteq m; \]
\[ m \text{ acts not in accordance with } m \subseteq m; \text{ i.e., } m \subseteq m; \]
\[ m \subseteq m. \]

(1. 1) We will establish the convention of regarding the human being (the locus of our study) as a member of the set of living beings. Human being, we will denote as \( p \), living being as \( f \) and the set of human beings as \( L \). By the first axiom, we obtain the proposition:

\[ p \mid f \in L \]

which reads "The human being as a living being is a member of the set of living beings". The implied assumption is that there is in \( L \), \( l \) other than \( p \) \((\tilde{p})\), or in other words, \( p \) is a distinguished \( l- \) although we shall not for the present define the distinction. It is therefore reasonable to assume that there exists a set \( P \) such that \( p \in L \) for all and only \( p \). It is easy to prove \( P \subseteq L \).

(1. 11) If \( p_1 \) is any member of the subset \( P \), there exists a neighbourhood \( f_1 \), a collection of members one of which is \( p_1 \). In case \( f_1 \) contains another member \( p_2 \in P \), such that \( p_1, p_2 \in f_1 \), we will assume \( p_1 \mid f_1 \subseteq f_1 \). We will call this \( f_1 \), then, the "immediate personal ring" of \( p_1 \). If, moreover, \( f_1 \cap f_2 \neq 0 \) (where \( p_2 \notin f_1 \) but \( p_2 \notin f_2 \)), meaning that the intersection of \( f_1 \) and \( f_2 \) is nonempty, or in other words that there is a part of \( f_1 \) and \( f_2 \) which is common to both \( f_1 \) and \( f_2 \), i.e.

\[ (p_1 \in f_1) \cap (p_2 \in f_2) + 0, \]

there exists a set \( f' \) such that \( f' = f_1 \cap f_2 \). The two neighbourhoods \( f_1 \) and \( f_2 \) of \( p_1 \) and \( p_2 \) respectively are not mutually exclusive, on assumption, and they overlap to some extent. Again it stands to reason that there is a family of all such \( f' \) of \( p_1(f_1), p_2(f_2), \ldots, p_n(f_n) \), namely \( F \). We can convince ourselves that no \( p \in F \) is an iso-