THEOREMS ON THE EXTREMAL DECOMPOSITION IN A FAMILY OF SYSTEMS OF DOMAINS OF VARIOUS TYPES

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A problem on extremal decomposition in a family \( \mathcal{D} \) of systems of domains of various types on a finite Riemann surface \( \mathcal{R} \) is studied. In contrast to the known cases, the family \( \mathcal{D} \) contains a system of bigons whose boundary arcs are asymptotically similar to logarithmic spirals with arbitrarily given slopes in neighborhoods of their vertices. The main result of this work is a full description of the extremal system of domains in the family \( \mathcal{D} \) in terms of the associated quadratic differential \( Q(z)dz^2 \) which is uniquely determined by a number of conditions. This differential has poles of the second order at distinguished points on \( \mathcal{R} \) with prescribed initial terms in the expansions of \( Q(z) \) with respect to local parameters representing these poles. Bibliography: 13 titles.

Dedicated to the 90th anniversary of G. M. Goluzin's birth

INTRODUCTION

By a problem of extremal decomposition we mean the question on the maximum of a functional which is defined on a family \( \mathcal{D} \) of systems of domains \( D_i \) associated with a family \( \mathcal{H} \) of homotopy classes \( H_i \) and is equal to a linear combination

\[
\sum_i \alpha_i^2 \mathcal{M}(D_i)
\]  

(1)

of functions of the domains \( D_i \) (the modules or reduced modules of the domains \( D_i \) associated with the classes \( H_i \)), where \( \alpha_i^2 \) are real parameters. The theory of problems of extremal decomposition was started by J. A. Jenkins [1], who proved that the problem on the maximum of functional (1) in a certain family \( \mathcal{D} \) of systems of domains consisting of doubly connected domains and quadrangles on a finite Riemann surface \( \mathcal{R} \) is equivalent to the module problem for the family \( \mathcal{H} \), and clarified the role of the associated quadratic differential in each one of the two extremal questions. The proof of existence of the associated quadratic differential in [1] uses the Schiffer–Spencer variational method for functionals on Riemann surfaces. These studies were continued by K. Strebel (see [2]) and H. Renelt [3]; in this connection, see the comments in [4]. In [4], Jenkins gave the proof of the result of [1] based only on the techniques of the method of the extremal metric.

Already in [1], Jenkins indicated the possibility of extending the result of [1] to the case where functional (1) contains the reduced modules \( \mathcal{M}(D_i, b_i) \) of simply connected domains \( D_i \) with respect to some points \( b_i \in D_i \). A simple proof for the case where \( \mathcal{R} \) is \( \mathbb{C} \) or a domain on \( \mathbb{C} \) was given in [5]; the theorems proved in [5] have found a number of applications.

Further results in this direction are connected with the notion of reduced module of a bigon with nonzero inner angles at the vertices [6, 7]. The theorem of extremal decomposition in a family of systems of domains on \( \mathbb{C} \) containing such bigons was presented in [6] and obtained some interesting applications. Recently, A. Yu. Solylinin [8] has obtained a theorem on extremal decomposition in a family of systems of domains of several types on a Riemann surface.

For the first time, the problem on extremal decomposition in a family of systems of domains on \( \overline{\mathbb{C}} \) containing bigons whose boundary arcs are similar to logarithmic spirals with arbitrarily given slopes in neighborhoods of the vertices was investigated in [9, 10], but the proofs of some assertions in [9, 10] were not given.
In the present work, we prove a theorem on extremal decomposition of a finite Riemann surface \( \mathcal{R} \), where an admissible family of systems of domains contains a system of bigons of the type indicated above. This completes the study of the question of existence of a quadratic differential having poles of the second order of each of the three possible types and satisfying prescribed topological and metrical conditions, and explains the role of this differential in the extremal problem considered.

§1. THE REDUCED MODULE OF A BIGON AND ITS PROPERTIES

Let us give the definition of the reduced module of a bigon with respect to its distinguished boundary elements \([6, 7, 9]\). Let \( D \subset \mathbb{C} \) be a simply connected domain of hyperbolic type with distinguished boundary elements \( b_1 \) and \( b_2 \) with supports at different or coinciding points \( b_1 \) and \( b_2 \). For the sake of definiteness, we assume that \( b_1, b_2 \in \mathbb{C} \) (the definitions given below can be directly extended to the general case). We will assume that the domain \( D \) satisfies the following condition (*).

Condition (*). Let \( \zeta = g(z) \) be the conformal homeomorphism of \( D \) onto the strip \( \Pi_0 = \{ \zeta : -1/2 < \text{Im} \zeta < 1/2 \} \) such that \( \text{Re} g(b_1) = -\infty \) and \( \text{Re} g(b_2) = +\infty \). Let \( \varepsilon_1 \) and \( \varepsilon_2 \) be sufficiently small positive numbers and let \( U(b_1, \varepsilon) = \{ z : |z - b_1| < \varepsilon \} \). Then, in the connected component \( \Delta_k(\varepsilon_k) \) of \( D \cap U(b_k, \varepsilon_k) \) with boundary element \( b_k \), we have the relation

\[
g(z) = (-1)^{k-1} \{ A_k e^{-i\beta_k \log(z - b_k)} + g_k(z) \}, \quad k = 1, 2,
\]

where \( A_k > 0, \beta_k \in (-\pi/2, \pi/2) \), \( g_k(z) \) is a regular function in \( \Delta_k(\varepsilon_k) \) such that

\[
g_k(z) = c_k + o(1) \quad \text{for} \quad z \to b_k, \ z \in \Delta_k(\varepsilon_k),
\]

and some fixed branch of \( \log(z - b_k) \) is considered.

It is clear that if a bigon \( D \) satisfies condition (*), then in a neighborhood of its distinguished boundary element \( b_k \) the sides of \( D \) are asymptotically like the trajectories of the quadratic differential

\[
Q_k(z)dz^2 = \frac{e^{-2i\beta_k \log(z - b_k)}}{(z - b_k)^2} \cdot
\]

These trajectories are rectilinear rays if \( \beta_k = 0 \) and logarithmic spirals with slope \( \beta_k \) if \( 0 < |\beta_k| < \pi/2 \).

Let \( \varepsilon_1 \) and \( \varepsilon_2 \) be sufficiently small positive numbers. Consider the quadrangles \( D(\varepsilon_1, \varepsilon_2) \) obtained from the bigon \( D \) by deleting some special neighborhoods of its boundary elements \( b_1 \) and \( b_2 \). The opposite sides of the quadrangles \( D(\varepsilon_1, \varepsilon_2) \) contained in the boundaries of the neighborhoods deleted are denoted by \( S_k(\varepsilon_k) \), \( k = 1, 2 \).

Let \( z_k^{(0)} \) and \( z_k^{(1)} \) be the points of intersection of the sides of \( D \) with the arc of the circle \( C_k(\varepsilon_k) = \{ z : |z - b_k| = \varepsilon_k \} \) lying on the boundary of \( \Delta_k(\varepsilon_k) \). Let \( \beta_k = 0 \). Then the arc of \( C_k(\varepsilon_k) \) connecting \( z_k^{(0)} \) and \( z_k^{(1)} \) lying on the boundary of \( \Delta_k(\varepsilon_k) \) is taken as \( S_k(\varepsilon_k) \). Let \( \beta_k \neq 0 \). Let \( S_k(z) \) be the orthogonal trajectory of differential (3) passing through \( z \). Let \( S_k(z_k^{(j)}, z_k^{(j)}) \), \( j = 0, 1 \), be the arc of \( S_k(\varepsilon_k) \) that connects \( z_k^{(j)} \) and \( z_k^{(j)} \) and is a connected component of \( D \cap S_k(z_k^{(j)}) \). Then we choose as \( S_k(\varepsilon_k) \) that one of the arcs \( S_k(z_k^{(j)}, z_k^{(j)}) \), \( j = 0, 1 \), which is lying on the closed circle \( \overline{U(b_k, \varepsilon_k)} \).

Let \( \varphi_k(\varepsilon_k) \) and \( \psi_k(\varepsilon_k) \) be the variations of \( \text{Arg}(z - b_k) \) on the arc of \( C_k(\varepsilon_k) \) lying on the boundary of \( \Delta_k(\varepsilon_k) \) and on the arc \( S_k(\varepsilon_k) \) going in the direction of increase of \( \text{Arg}(z - b_k) \), respectively. It follows from condition (*) that

\[
\varphi_k(\varepsilon_k) = \varphi_k + o(1); \quad \psi_k(\varepsilon_k) = \psi_k + o(1),
\]

where

\[
\psi_k = \varphi_k \cos^2 \beta_k = \frac{\cos \beta_k}{A_k}, \quad k = 1, 2.
\]