THE HOMOLOGY OF GRADED INFINITE-DIMENSIONAL LIE ALGEBRAS IN CONNECTION WITH MACDONALD IDENTITIES

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D. Fuchs' monograph devoted to the cohomology of infinite-dimensional Lie algebras contains an error in calculating the homology of a graded affine Kac-Moody algebra of type $A_1^{(1)}$, so that the proof of the corresponding Macdonald identity, which is based on that calculation, is incorrect. In the present paper, a revised proof is suggested. Bibliography: 4 titles.

The purpose of the present paper is to correct the proof given in [4] of the simplest Macdonald identity. The identity is of the form

$$\prod_{r=1}^{\infty} \left( 1 - t_1^r t_2^r \ldots t_n^r \right) \prod_{1 \leq i < j \leq n} \left( 1 - t_i^{r-1} t_{i+1}^{r-1} \ldots t_j^{r-1} \ldots t_n^{r-1} \right)$$

$$\times \prod_{1 \leq i < j \leq n} \left( 1 - t_i^r t_j^{r-1} t_{i-1}^{r-1} \ldots t_{i-1}^{r-1} t_i^{r-1} \ldots t_n^{r-1} \right) = \sum \pm t_1^{k_1} \ldots t_n^{k_n},$$

where $n \geq 2$; the sum on the right-hand side is taken over the multiindices $(k_1, \ldots, k_n) \in \mathbb{Z}_+^n$ that are the roots of the quadratic polynomial

$$Q(k_1, \ldots, k_n) = \sum_{i=1}^{n} k_i - \frac{1}{2} \sum_{i=1}^{n} (k_i - k_{i-1})^2$$

(henceforth, we assume that $k_0 = k_n$) and satisfy the following condition:

the integers $k_i - k_{i-1} + i - 1$ (i = 1, 2, ..., n) are distinct modulo $n$. (*)&

The sign of the monomial $t_1^{k_1} \ldots t_n^{k_n}$ in the sum on the right corresponds to the parity of the permutation

$$((k_1 - k_n) \mod n, (k_2 - k_1 + 1) \mod n, \ldots, (k_n - k_{n-1} + n - 1) \mod n)$$

of the numbers $(0, 1, 2, \ldots, n - 1)$. The original version of the proof [4], which uses the homology of graded Lie algebras, contains an error in the computation of the homology of a certain algebra $T^+(n)$, and, as a consequence, an incompleteness in the statement of the identity: condition (*) is lacking. Without (*), formula (1) from [4, §3.2.3] is incorrect for $n \geq 5$ (see the Appendix). However, we emphasize that the present paper does not pretend to revise the plan of the proof given in [4] or the devices of the homology theory involved. The only goal of the paper is to complete the computation of the homology of algebras in question; this will allow us to prove a correct version of (1).

Originally, the Macdonald identities were discovered by I. Macdonald [3], who generalized the Weyl formula [N. Bourbaki, Groupes et algèbres de Lie, Chaps. IV, V, VI, Hermann, Paris, 1968] to the case of affine root systems. V. Kac [1] demonstrated that the identities result from the theory of representations of the Kac–Moody algebras; one can find a proof of Macdonald identities based on the calculation of the character of the trivial representation of a Kac–Moody algebra, e.g., in [2, §12.1]. The proof in the context of homology theory, considered in [4] and in the present paper, is another approach to the Macdonald identities. We note that (1) is a special case of the Macdonald identities. In terms of [2], this case corresponds to the
Kac–Moody algebras of type $A^{(1)}_{n-1}$. Formula (1) involves summation over the nodes of an integer lattice, whereas in a more general situation [2], an additional summation over the elements of the Weyl group is required. In the case $n = 2$, Eq. (1) is equivalent to the famous Gauss–Jacobi identity for a triple product.

For other approaches to the proof of the Macdonald identities, see the bibliography in [2].

The proof of (1) is based on the standard scheme of producing combinatorial identities with the help of the homology of graded Lie algebras (see [4, §3.2.2]). Namely, let a Lie algebra $g$ be graded: $g = \bigoplus_{\alpha \in A} g(\alpha)$, where $A$ is a commutative semigroup written additively, and $[g(\alpha), g(\beta)] \subseteq g(\alpha + \beta)$. Assume that $d_\alpha = \dim g(\alpha) < \infty$.

Let $C_q^{(\alpha)} = \{x_1 \wedge \cdots \wedge x_q \mid x_i \in g(\alpha), \sum_{i=1}^q \alpha_i = \alpha\}$ be homogeneous components of the chain spaces; they form, together with (the usual) differential $\partial$, a graded chain complex. Let $h_q^{(\alpha)}$ denote the dimension of its graded homology $H_q^{(\alpha)}$. Then the identity

$$\prod_{\alpha}(1 - e^\alpha)^{d_\alpha} = \sum_{q \geq 0, \alpha} (-1)^q h_q^{(\alpha)} e^\alpha$$

holds, where the symbols $e^\alpha$ are formal exponents of elements of $A$. If the graded algebra $g$ is fixed, then the parameters $d_\alpha$ and $h_q^{(\alpha)}$ are integers, and (2) is converted to an identity, which enables us to represent the infinite product as a series.

Macdonald identity (1) can be obtained from (2) by taking $\mathbb{Z}_n^+ = \{(\alpha_1, \ldots, \alpha_n) \mid \alpha_i \in \mathbb{Z}, \alpha_i \geq 0\}$ as $A$, specifying $e^\alpha = t_1^{\alpha_1} \cdots t_n^{\alpha_n}$ for $\alpha \in A$, and by regarding the graded algebra $\mathcal{T}^+(n)$ defined below as $g$.

Let $\mathcal{T}(n) = (\mathfrak{gl}(n, \mathbb{C})^{S^1})^{\text{pol}} = \mathfrak{gl}(n, \mathbb{C}) \otimes \mathbb{C}[t, t^{-1}] = \bigoplus_{\tau \in \mathbb{Z}} (\mathfrak{gl}(n, \mathbb{C}) t^\tau)$ be the algebra consisting of the mappings from $S^1$ to $\mathfrak{gl}(n, \mathbb{C})$ defined by trigonometric polynomials, with pointwise commutator, or, what is the same, the algebra of Laurent polynomials with coefficients in $\mathfrak{gl}(n, \mathbb{C})$. Now we introduce a grading in $\mathcal{T}(n)$. Let $e_{ij}^\tau$ be the matrix with 1 at the intersection of the $i$th row and the $j$th column, and zeros at the other positions; we assign to $e_{ij}^\tau t^\sigma$ ($\tau \in \mathbb{Z}$), an element of the basis in $\mathcal{T}(n)$, the weight in accordance with the following rule:

- $(r, \ldots, r, \tau)$ if $i = j$;
- $(r, \ldots, r, r + 1, \ldots, r + 1, r, \ldots, r)$ if $1 \leq i < j \leq n$;
- $(r, \ldots, r, r - 1, \ldots, r - 1, r, \ldots, r)$ if $1 \leq j < i \leq n$.

The linear span of the basis elements of weight $\beta \in \mathbb{Z}^n$ is the homogeneous component $\mathcal{T}(n)^{(\alpha)}$ of the algebra $\mathcal{T}(n)$, corresponding to the weight $\beta \in \mathbb{Z}^n$. The condition $[\mathcal{T}(n)^{(\alpha)}, \mathcal{T}(n)^{(\beta)}] \subseteq \mathcal{T}(n)^{(\alpha + \beta)}$ is verified directly.

Let $T^+(n)$ be a subalgebra of $\mathcal{T}(n)$:

$$T^+(n) = N^+(\mathfrak{sl}(n, \mathbb{C})) \oplus \bigoplus_{r \in \mathbb{Z}, r \geq 1} \mathfrak{sl}(n, \mathbb{C}) t^r$$

(here $N^+(\mathfrak{sl}(n, \mathbb{C}))$ is the subalgebra of $\mathfrak{sl}(n, \mathbb{C})$ that consists of the upper triangular matrices with zero diagonal). The grading in $T^+(n)$ is induced from $\mathcal{T}(n)$:

$$T^+(n)^{(\alpha)} = T^+(n) \cap \mathcal{T}(n)^{(\alpha)}.$$