C-ALGEBRAS AND ALGEBRAS IN PLANCHEREL DUALITY

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For an arbitrary (possibly noncommutative) C-algebra, a positivity condition generalizing the Krein condition for a commutative case is defined. We show that the class of positive C-algebras includes those arising in algebraic combinatorics from association schemes (possibly noncommutative). It is proved that the category of positive C-algebras is equivalent to the category of pairs of algebras in Plancherel duality, one of which is commutative. Bibliography: 7 titles.

1. INTRODUCTION

In this paper, the connection between C-algebras, representing one of the main concepts of algebraic combinatorics, and an important class of bialgebras, namely, algebras in Plancherel duality, is established. We consider these notions as they were defined; a further development of them, as well as a more detailed exposition of the above connection, will be given in a separate paper.

The association schemes investigated in algebraic combinatorics lead to matrix algebras having a linear basis with special properties. This basis actually determines a commutative multiplication (more correctly a comultiplication) and, consequently, a bialgebra, which is a linear space with multiplication and comultiplication compatible with each other. A natural question arises on how to describe this structure in abstract algebraic terms. The advantage of such a description consists of the possibility of constructing (and subsequently applying) general representation theory. The natural category arising in this way can include objects which have no matrix representations mentioned; the problem of the existence of combinatorial objects (such as association schemes or block designs) then transforms into the problem of searching for special representations of appropriate algebraic systems.

Attempts at such constructions have resulted in the concept of a C-algebra (see [1] and Sec. 2 of this paper). However, as is shown in the paper, this concept is not quite adequate in algebraic combinatorics: the axioms of C-algebra do not include the natural condition of positivity which is inherent in all combinatorial algebras. It should be noted that whereas for commutative C-algebras this additional condition (the positivity of the structure constants and the Krein constants) can easily be written down (see [1]), in the general case the situation is more complicated.

In 1972, one of the authors (see [3]), in generalizing the duality theory of Krein–Tanaka, Hopf, and others, proposed a new, weaker (than in the definition of Hopf algebras), coordination of two multiplications (in other terms, multiplication and comultiplication) in dual spaces; namely, the concept of algebras in positive duality. In [4, 5], S. V. Kerov studied this concept and, in particular, its finite-dimensional version (which is basically necessary for combinatorics). The analog of the duality theory for groups with invariant measure led him to the concept of algebras in Plancherel duality, or Plancherel triples (see Sec. 3). It is this concept that has proved to be more adequate for the objects studied in algebraic combinatorics. There are numerous examples of such triples coming from groups and algebras.

The main result of the paper is of the following statement.

**Theorem 1.** The categories of positive C-algebras and pairs of algebras in Plancherel duality, one of which (for instance, the second) is commutative, are equivalent.

In this connection, we note that the axiomatics of algebras in Plancherel duality does not require the commutativity of any of the algebras. Up to now, only a few interesting examples of the double noncommutative case (such as finite quantum groups) are known. However, the study of this case seems to be highly informative: it can be viewed as the quantum deformation of classical combinatorics. Therefore, in our

opinion, algebras in Plancherel duality (in the most general noncommutative variant) represent a natural algebraic equivalent of association schemes.

One of the main problems of algebraic combinatorics consists of the characterization of $C$-algebras coming from association schemes. The positivity hypothesis of Theorem 1 gives a necessary condition of the existence. Indeed, the adjacency algebra of an association scheme is a special case of a homogeneous cellular algebra (see Sec. 2). To such an algebra $A \subseteq \text{Mat}_n(C)$, we associate a triple $(A, B, \langle , \rangle)$ where $B$ is the $*$-algebra coinciding with $A$ as a linear space and equipped with the coordinate-wise multiplication, with respect to the standard basis of $A$, and the complex conjugation as an involution. The bilinear form $\langle , \rangle$ is defined by $\langle a, b \rangle = \frac{1}{n} \text{Tr}(ab^T)$. The proof of the following statement immediately follows from Theorem 1 and Proposition 2.6 below.

**Theorem 2.** A triple $(A, B, \langle , \rangle)$ is Plancherel for any homogeneous cellular algebra $A$.

Another application of the objects discussed concerns multivalued involutory groups (see [2]).

### 2. POSITIVE C-ALGEBRAS

Let $A$ be an associative algebra over $C$ with unit 1, a semilinear involutory antiautomorphism $*$, and a special linear basis $R$. The structure constants of $A$ relative to the basis $R$ are defined by the relation

$r \cdot s = \sum_{t \in R} c_{r,s}^t \cdot t$, where $r, s \in R$.

**Definition 2.1.** The pair $(A, R)$ is called a C-algebra if the following conditions are satisfied.

(C1) $1 \in R$ and $R^* = R$.

(C2) The structure constants $c_{r,s}^t$ of the algebra $A$ relative to the basis $R$ are real numbers.

(C3) The equality $c_{r,s}^t = d(r) d_{r,s}$ holds for all $r, s \in R$, where $d(r) > 0$ and $d$ is the Kronecker symbol.

(C4) The linear functional on $A$ defined by the map $r \mapsto d(r), r \in R$, is a one-dimensional $*$-representation of the algebra $A$. In particular, $d(r) = d(r^*)$ for all $r \in R$.

**Remark 2.2.** In algebraic combinatorics, the concept of a C-algebra is used basically for commutative algebras (see [1]).

By an isomorphism of C-algebras we mean a $*$-isomorphism of the underlying algebras which preserves the bases.

We define in $A$ a coordinate-wise multiplication $\circ$ with respect to the basis $R$ by

$r \circ s = \delta_{r,s} r, \quad r, s \in R$. (1)

Evidently, this turns the linear space $A$ into a commutative associative algebra over $C$ with unit $J = \sum_{r \in R} r$.

**Definition 2.3.** A C-algebra $A = (A, R)$ is called positive if

(P1) all structure constants $c_{r,s}^t$ of the algebra $A$ with respect to the basis $R$ are nonnegative reals;

(P2) the cone $A^+ = \{aa^* : a \in A\}$ of all nonnegative elements of the algebra $A$ is closed under the multiplication $\circ$.

**Remark 2.4.** The following example shows that condition (P2) does not follow from condition (P1). In the commutative C-algebra $(A, \{1, r, s\})$ with the multiplication

$r^2 = m \cdot 1 + (m - 1) \cdot s$

$s^2 = m \cdot 1 + r + (m - 2) \cdot s$

$rs = (m - 1) \cdot r + s$

and the involution $r^* = r, \quad s^* = s$, condition (P1) is obviously satisfied for all $m \geq 2$. On the other hand, condition (P2) is violated for each $m > 2$. (Probably, such examples were known earlier.)

Unfortunately, the term C-algebra is extremely unsuccessful, since it can be mixed with the terms C*-algebra and C-algebra.